Group representations in Banach spaces and Banach lattices

Proefschrift

ter verkrijging van
de graad van Doctor aan de Universiteit Leiden,
op gezag van de Rector Magnificus
prof. mr. P.F. van der Heijden,
volgens besluit van het College voor Promoties
te verdedigen op woensdag 18 april 2012
klokke 16.15 uur

door

Marten Rogier Wortel

geboren te Delft
in 1981

Samenstelling van de promotiecommissie:

promotor:	Prof. dr. S.M. Verduyn Lunel
copromotores:	Dr. M.F.E. de Jeu Prof. dr. B. de Pagter (Technische Universiteit Delft)
Overige leden:	Dr. O.W. van Gaans Prof. dr. N.P. Landsman (Radboud Universiteit Nijmegen) Prof. dr. P. Stevenhagen Prof. dr. A.W. Wickstead (Queen's University Belfast)

Group representations in Banach spaces and Banach lattices

Contents

Chapter 1

Introduction

Motivated by quantum mechanics, amongst others, where there are many examples of group representation in Hilbert spaces, strongly continuous unitary representations of locally compact groups have been studied extensively. In particular, the decomposition theory is now well developed, which will now be illustrated by an example. Consider the group $[0, 2\pi)$ with addition modulo 2π, which is isomorphic to the circle group S^1, and the Hilbert space $L^2([0, 2\pi))$. We examine the left regular representation ρ of the group $[0, 2\pi)$ in $L^2([0, 2\pi))$ defined by $(\rho_s f)(x) := f(x - s)$, for $f \in L^2([0, 2\pi))$ and $s, x \in [0, 2\pi)$. The collection of functions $\{e^{in\cdot} := x \mapsto e^{inx}\}_{n\in\mathbb{Z}} \subset L^2([0, 2\pi))$ forms an orthogonal basis of $L^2([0, 2\pi))$, and the decomposition of a function $f \in L^2([0, 2\pi))$ into this orthogonal basis is its Fourier series $f = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in\cdot}$, where

$$\hat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx}\, dx = \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{e^{inx}}\, dx \tag{1.1}$$

denotes the n-th Fourier coefficient. Then, by the properties of the Fourier series, we obtain for $s \in [0, 2\pi)$,

$$\rho_s \left(\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in\cdot} \right) = \sum_{n\in\mathbb{Z}} e^{-ins} \hat{f}(n)e^{in\cdot}. \tag{1.2}$$

If we fix $n \in \mathbb{Z}$, then the one dimensional subspace spanned by $e^{in\cdot}$ is invariant under ρ, and on this subspace the operator ρ_s is just a pointwise multiplication by e^{-ins}. In particular, the restriction of ρ to this subspace is an irreducible representation. Hence ρ is an orthogonal direct sum over $n \in \mathbb{Z}$ of irreducible representations. This example is a special case of the unitary representation theory of compact groups, which states that for any strongly continuous unitary representation of a compact group, the representation splits as an orthogonal direct sum of finite dimensional irreducible representations.

For non-compact locally compact groups such a direct sum decomposition is not always possible, but it is still possible to view the original representation as somehow "built up" from irreducible ones. The following example, which is similar to the above example, explains how this is done. Let G be any abelian locally compact group with a Haar measure μ, i.e., a left invariant regular measure on G which is finite on compact sets. Again we consider the left regular representation ρ of G on $L^2(G,\mu)$ defined by $(\rho_s f)(r) := f(r-s)$, for $s, r \in G$. Consider the dual group $\Gamma = \mathrm{Hom}(G, S^1)$. The dual group, equipped with the compact-open topology, is again a locally compact abelian group. An example is $G = \mathbb{R}$ with its natural topology, then $\Gamma \cong \mathbb{R}$ with its natural topology. The Fourier transform $f \mapsto \hat{f}$ from $L^1(G,\mu)$ to $C_0(\Gamma)$ defined by

$$\hat{f}(\gamma) := \int_G f(r)\overline{\gamma(r)}\, d\mu(r), \quad \gamma \in \Gamma$$

is a generalization of (1.1), and its restriction to $L^1(G,\mu) \cap L^2(G,\mu)$ maps isometrically into $L^2(\Gamma,\lambda)$, where λ is an appropriately chosen Haar measure on Γ, and it extends to an isometric isomorphism of Hilbert spaces between $L^2(G,\mu)$ and $L^2(\Gamma,\lambda)$. So we may transport our representation ρ from $L^2(G,\mu)$ to $L^2(\Gamma,\lambda)$, and there we obtain, using the properties of the Fourier transform, for $f \in L^2(G,\mu)$, $s \in G$ and almost every $\gamma \in \Gamma$,

$$(\rho_s \hat{f})(\gamma) = \overline{\gamma(s)}\hat{f}(\gamma),$$

which is similar to (1.2); here the representation corresponds to a pointwise almost everywhere multiplication by $\overline{\gamma(s)}$, some sort of "integral" of pointwise multiplications, in particular, of irreducible representations. This can be formalized using the notion of direct integrals of Hilbert spaces and direct integrals of representations, and using this notion, we can state the main theorem on decomposing strongly continuous unitary representations of locally compact groups in terms of irreducible representations, cf. [50, Corollary 14.9.5].

Theorem 1.1. *Let G be a separable locally compact group, H a separable Hilbert space and ρ a strongly continuous unitary representation of G on H. Then H is isometrically isomorphic to a direct integral of Hilbert spaces, such that under this isomorphism, the representation ρ corresponds to a direct integral of irreducible representations.*

Unitary representations often arise naturally whenever a group acts on a set, e.g., let $X \subset \mathbb{C}$ be the closed unit disc with Lebesgue measure, then S^1 acts naturally on X and a strongly continuous unitary representation ρ of S^1 in $L^2(X)$ is defined by $(\rho_s f)(x) := f(s^{-1}x)$, for $s \in S^1$, $f \in L^2(X)$ and $x \in X$. By the above such representations can be decomposed into irreducibles. However, the same formula yields a strongly continuous representation of S^1 in the Banach spaces $L^p(X)$, for $1 \leq p < \infty$, and in $C(X)$. Hence such representations on Banach spaces occur naturally - what about these representations? Do we have a similar decomposition theory as in the unitary case?

This thesis is a contribution to the theory of such representations. It consists of two parts. The first part, Chapter 2, is about the crossed product. The second part, Chapters 3 and 4, is about positive representations in partially ordered vector spaces. We will now discuss the first part.

Crossed products

When studying group representations, it is often useful to look at algebras. An example is the group algebra $k[G]$ of a finite group G over a field k. This algebra has the property that there is a bijection between representations of G on k-vector spaces and algebra representations of $k[G]$ on the same vector space, and so questions about group representations can be translated into questions about algebra representations.

In the theory of unitary representations, such an algebra also exists. Given a locally compact group G, this object is the group C^*-algebra $C^*(G)$, a C^*-algebra for which the nondegenerate algebra representations in a Hilbert space are in bijection with the strongly continuous unitary representations of the group in that Hilbert space. The C^*-algebra $C^*(G)$ is a crucial tool in proving Theorem 1.1. Indeed, all the hard work lies in proving a similar fact about representations of C^*-algebras on Hilbert spaces, and then the result about unitary representations follows immediately by applying this to $C^*(G)$.

In view of the above it is desirable, given a locally compact group G, to have a Banach algebra such that some of its algebra representations in certain Banach spaces are in bijection with some of the strongly continuous group representations of G in the same class of Banach spaces. The reason for not considering all representations is the following. In the Hilbert space case, there is up to isomorphism only one separable infinite dimensional Hilbert space. However, there is a great diversity of infinite dimensional separable Banach spaces, and to consider all representations in all these spaces seems a daunting task. It would be much better if the above Banach algebra can be specialized to specific situations. E.g, if one is interested in strongly continuous isometric group representations in L^p-spaces for some $p \geq 1$, then one would want a Banach algebra such that its algebra representations in L^p-spaces are in bijection with these group representations. Or, if one is interested in uniformly bounded representations in spaces of continuous functions, then one would want a different Banach algebra with a bijection concerning these representations in these spaces. In other words, the construction of the group C^*-algebra needs to be generalized such that it can be adapted to whatever representations one is interested in, instead of only the unitary group representations.

The group C^*-algebra is a special case of a more general object called a crossed product C^*-algebra, which is not only useful in translating unitary group representations to algebra representations, but also has applications concerning induced representations of subgroups. Hence we will generalize the crossed product C^*-algebra, which we will now briefly discuss.

A *C^*-dynamical system* is a triple (A, G, α), where A is a C^*-algebra, G is a locally compact group and $\alpha\colon G \to \mathrm{Aut}(A)$ is a strongly continuous action of G on A. This can be viewed as a generalization of a locally compact group, since if $A = \mathbb{C}$, the action α has to be trivial and G is the only nontrivial object. A *covariant representation* of (A, G, α) is a pair (π, U), where π is a representation of

A in a Hilbert space, and U is a strongly continuous unitary representation of G in the same Hilbert space, satisfying the covariance relation

$$U_s \pi(a) U_s^{-1} = \pi(\alpha_s(a)),$$

for all $s \in G$ and $a \in A$. Again, if $A = \mathbb{C}$, π is trivial and hence the covariant relation is always satisfied, and the group representation is the only nontrivial object. Hence studying covariant representations of $(\mathbb{C}, G, \alpha)$ is the same as studying strongly continuous unitary representations of G. The crossed product C^*-algebra is a C^*-algebra $A \rtimes_\alpha G$ with the property that the class of covariant representations of (A, G, α) is in bijection with the class of nondegenerate representations of $A \rtimes_\alpha G$ in Hilbert spaces. If $A = \mathbb{C}$, we recover the group C^*-algebra $C^*(G)$.

The above can be generalized to the Banach algebra case as follows. A Banach algebra dynamical system is a triple (A, G, α) with the same properties as a C^*-dynamical system, except that A is only assumed to be a Banach algebra. Covariant representations are generalized by allowing the representations to be in Banach spaces instead of Hilbert spaces. The main difference is in the class of covariant representations being considered; in the Hilbert space case, this class equals *all* covariant representations in Hilbert spaces. Since, as explained earlier, we want to vary the class of covariant representations being considered, this class is an additional variable, which will be called $\mathcal{R}$, going into the crossed product construction. It turns out that one cannot consider all classes $\mathcal{R}$. There has to be some uniform bound on the norm of the algebra representations, and the norm of the group representations has to be uniformly bounded by some fixed function $\nu \colon G \to [0, \infty)$ which is bounded on compact sets, i.e., $\|U_r\| \leq \nu(r)$ for all $(\pi, U) \in \mathcal{R}$. Note that these requirements are automatically satisfied in the C^*-algebra case, as C^*-algebra representations in Hilbert spaces are contractive and unitary representations are isometric. Given such a class $\mathcal{R}$, one also needs to define the class of $\mathcal{R}$-continuous covariant representations (Definition 2.5.1), which is in general a larger class than $\mathcal{R}$. In the C^*-algebra case, these classes coincide.

A technical obstacle that we encountered while generalizing the crossed product C^*-algebra is as follows. In the C^*-algebra case, it is at some point necessary to extend a bounded nondegenerate representation of a C^*-algebra to its multiplier algebra, which can be done easily using C^*-theory. The Banach algebra analogue of this is, given a nondegenerate bounded representation of a Banach algebra, to obtain an extension of this representation to centralizer algebras of the original Banach algebra. It turns out that this can be done in a satisfactory manner, which was worked out in [9]. After overcoming this technical obstacle, and incorporating the new features concerning the $\mathcal{R}$-continuous covariant representations, it turns out that the crossed product C^*-algebra can indeed be generalized, cf. Theorem 2.8.1, the main result of Chapter 2. It states that, given a Banach algebra dynamical system with a mild condition on A (it has to have a bounded approximate left identity), a class $\mathcal{R}$ as above and a class $\mathcal{X}$ of Banach spaces, there is a Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ such that its bounded nondegenerate algebra representations in spaces from $\mathcal{X}$ are in bijection with the class of $\mathcal{R}$-continuous covariant representations of (A, G, α) in spaces from $\mathcal{X}$.

We have specialized the above result to certain natural classes of group representations in Banach spaces, cf. Section 2.9, and this allows us to study such group representations by studying the representations of the specific Banach algebras thus obtained instead. Further investigation of these special Banach algebras is needed to optimally exploit this result, but at least the problem has become more tractable now from a functional analytical point of view, since a Banach algebra has more functional analytic structure than a group. Given the success of the archetypical transition from the group to the group C^*-algebra in the case of unitary group representations, the availability of such a transition, "tuned to the situation at hand", can be considered as a step forward towards a decomposition theorem for other classes of group representations than the unitary ones.

Another type of group representations one would like to understand better are the positive group representations, which we will discuss in the next section. For such representations, the appropriate Banach algebra of crossed product type will have to be a so-called Banach lattice algebra, and the construction of the crossed product needs further modification. We leave this for further research, noting that the results and techniques in Chapter 2 provide a concrete model to start from.

For unitary representation of compact groups, however, it is already possible to obtain the existence of a decomposition into irreducibles without the use of the group C^*-algebra. One might hope that a similar phenomenon occurs for positive representations of compact groups. As is shown in Chapters 3 and 4, which constitute the second part of this thesis, for certain spaces this is indeed the case.

We will now specialize our discusssion from general Banach space representations to positive representations.

Positive representations

We return again to our motivating example of the representation ρ of S^1 in $L^p(X)$ $(1 \leq p < \infty)$ and $C(X)$, where $X \subset \mathbb{C}$ is the closed unit disc, defined by

$$(\rho_s f)(x) := f(s^{-1}x),$$

for $s \in S^1$ and $x \in X$. It is clear that, for $s \in S^1$, the maps ρ_s are positive, i.e., they map positive functions to positive functions. This positivity of the operators ρ_s is the context for Chapter 3 and Chapter 4.

The natural partial order on functions spaces such as $L^p(X)$, i.e., $f \geq g$ if and only if $f(x) \geq g(x)$ for almost every x, can be studied by the following abstraction. A *partially ordered vector space* is a real vector space equipped with a partial order that is compatible with the vector space structure, i.e., if the vectors x and y are positive, then $x + y$ is positive, and if λ is a positive scalar, then λx is positive. A *Riesz space* is a partially ordered vector space where each pair of vectors x and y has a supremum $x \vee y$ and an infimum $x \wedge y$. In a Riesz space one can define an absolute value as in function spaces, by $|x| := x \vee (-x)$, and x and y are called *disjoint* if $|x| \wedge |y| = 0$. If L is a subset of a Riesz space, then L^d denotes the disjoint complement of A, i.e., all vectors that are disjoint from all vectors from

A. A *Banach lattice* is a Riesz space, and a Banach space, such that the norm is compatible with the order structure, i.e., if x and y are vectors such that $|x| \leq |y|$, then $\|x\| \leq \|y\|$. Many function spaces considered in analysis, such as L^p-spaces and spaces of continuous functions, are Banach lattices.

By the above it makes sense to study positive representations in Banach lattices, as they appear naturally: whenever there is a group acting on some set, more often than not there is an induced positive representation in a Banach lattice of functions defined on that set. Very little is known about positive representations. The natural question in this case, similar to the unitary case, is whether a positive representation in a Banach lattice can be decomposed into order indecomposable subrepresentations. This will be the main theme in our investigations of positive representations.

In Chapter 3 we consider the simplest case: the finite dimensional case. Since vector space topologies are not interesting in the finite dimensional setting, we look at the more general setting of Riesz spaces, instead of Banach lattices. We are interested in decompositions, and a natural question is whether an order indecomposable positive representation of a finite group is finite dimensional. With an order indecomposable positive representation we mean a positive representations such that the Riesz space cannot be written as the order direct sum of two subspaces that are both invariant under the representation. An order direct sum of a Riesz space E means a direct sum $E = L \oplus M$, such that whenever $x = y + z \in E$ is positive, with $y \in L$ and $z \in M$, then y and z are positive. In this case L and M are automatically so-called projection bands which are each other's disjoint complement, so $E = L \oplus L^d$. This is similar to the Hilbert space case, where a closed linear subspace L of a Hilbert space H induces an orthogonal decomposition $H = L \oplus L^\perp$. Moreover, if a projection band is invariant under a positive representation of a group, its disjoint complement is also invariant, which is similar to the Hilbert space case where the orthogonal analogue holds for a unitary representation of a group. This easily implies that order indecomposability of a positive representation in a Riesz space is equivalent with its projection band irreducibility, i.e., that it does not possess a nontrivial proper invariant projection band. Again, this is similar to the case of unitary representations, where indecomposability is equivalent with irreducibility, i.e., with the absense of nontrivial proper invariant closed subspaces. Hence the above question can be reformulated as: is a projection band irreducible positive representation of a finite group finite dimensional?

The corresponding question in the unitary case is trivially true. Indeed, let ρ be an irreducible unitary representation of a finite group in a Hilbert space H. Take a nonzero vector $x \in H$ and consider the subspace generated by its orbit under ρ. This subspace is finite dimensional and hence closed, and by construction ρ-invariant, so it must equal the whole space H, hence H is finite dimensional. Unfortunately this proof breaks down in the ordered setting, as bands, in particular projection bands, are generally infinite dimensional, e.g., in $L^p([0,1])$, for any p, all nontrivial bands are infinite dimensional.

In the ordered case, the above question has a negative answer. Indeed, the representation of the trivial group in $C([0,1])$ is projection band irreducible. In this example the cause lies with the Riesz space, one might say, as $C([0,1])$ does not have any proper nontrivial projection bands at all: it is far from being what is called Dedekind complete. All L^p-spaces, on the other hand, are Dedekind complete.

If we assume that the Riesz space is Dedekind complete, then the situation improves. In this case we managed to show, with an unusual proof based on induction on the order of the group, that if a positive representation of a finite group in a Dedekind complete Riesz space is projection band irreducible, then the Riesz space is finite dimensional, cf. Theorem 3.3.14. In this theorem we actually prove a little bit more, but this is the most important consequence.

Having obtained this result, we then looked at the positive representations of finite groups in finite dimensional Archimedean Riesz space. These spaces are order isomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$ with pointwise ordering. We first study the automorphism group of $\mathbb{R}^n$, i.e., the group of positive matrices with positive inverses, and show that it equals the semidirect product of the subgroup of diagonal matrices with strictly positive entries on the diagonal, and the subgroup of permutation matrices. Using some basic group cohomology methods, it turns out that every finite group of positive matrices equals a group of permutation matrices conjugated by a single diagonal matrix with strictly positive diagonal elements, and from this it follows easily that every positive representation equals a permutation representation conjugated by such a diagonal matrix. From this we obtain a nice characterization of the order dual of a finite group, i.e., the space of order equivalence classes of irreducible positive representations, in terms of group actions on finite sets, cf. Theorem 3.4.10, which is as follows.

Theorem 1.2. *Let G be a finite group. If $H \subset G$ is a subgroup, let E_H be the $|G : H|$-dimensional vector space of real-valued functions on G/H, equipped with the pointwise ordering. Let π^H be the canonical positive representation of G in E_H, corresponding to the action of G on G/H. Then, whenever H_1 and H_2 are conjugate, π^{H_1} and π^{H_2} are order equivalent, and the map*

$$[H] \mapsto [\pi^H]$$

is a bijection between the conjugacy classes of subgroups of G and the order equivalence classes of irreducible positive representations of G in nonzero finite dimensional Archimedean Riesz spaces.

Additionally, we obtain a unique decomposition of positive representations of finite groups in finite dimensional Archimedean Riesz spaces into band irreducibles.

We also show that characters do not, in general, determine representations, in the sense that there even exist band irreducible positive finite dimensional representations of finite groups, having the same character, which are not order isomorphic. Finally, we look at induction in the ordered setting, the categorical aspects of which are largely the same as in the nonordered setting, but for which the multiplicity version of Frobenius reciprocity turns out not to hold.

In Chapter 4 we take the above results to the next level: that of compact groups. As the image of a strongly continuous representation of a compact group in a Banach lattice is a group of positive operators which is compact in the strong operator topology, such compact groups of positive operators are investigated. We assume that these groups are contained in the product, which again is a semidirect product, of the group of central lattice automorphisms and the group of isometric lattice automorphisms. This is motivated by the above results on representations in $\mathbb{R}^n$ equipped with any of the p-norms, where the isometric lattice automorphisms are the permutation matrices and the central lattice automorphisms are the diagonal matrices with strictly positive elements on the diagonal, and so the whole automorphism group of $\mathbb{R}^n$ equals this semidirect product and hence every subgroup satisfies this assumption. This characterization of the automorphism group is also satisfied in many natural sequence spaces and spaces of continuous functions. However, not every Banach lattice has such a nice characterization of the automorphism group.

Under the additional technical Assumption 4.3.3 which is satisfied in many natural sequence spaces and spaces of continuous functions, we are able to obtain, as in the finite dimensional case, that such a compact group equals a group of isometric lattice automorphisms conjugated by a single central lattice automorphism. This is especially useful in the aforementioned spaces, as we have a nice description available of both the isometric lattice automorphisms and the central lattice automorphisms. Again this leads to a similar description of strongly continuous positive representations of compact groups with range in this product: it is a strongly continuous isometric positive representation conjugated by a single central lattice automorphism. Since everything depends only on the compactness in the strong operator topology of the image of the representation, we have the same result for arbitrary representations of arbitrary groups with compact image. Applying these results to the sequence space case, we obtain the following ordered analogue of the aforementioned theorem on the decomposition of strongly continuous unitary representations of compact groups, which is as follows, cf. Theorem 4.5.7.

Theorem 1.3. *Let E be a normalized symmetric Banach sequence space, let G be a group and let ρ be a positive representation of G in E. Then E splits into band irreducibles, in the sense that there exists an α with $1 \leq \alpha \leq \infty$ such that the set of invariant and band irreducible bands $\{B_n\}_{1\leq n\leq\alpha}$ (if $\alpha < \infty$) or $\{B_n\}_{1\leq n<\infty}$ (if $\alpha = \infty$) satisfies*

$$x = \sum_{n=1}^{\alpha} P_n x \quad \forall x \in E, \tag{1.3}$$

where $P_n \colon E \to B_n$ denotes the band projection, and the series is unconditionally order convergent, hence, in the case that E has order continuous norm, unconditionally convergent.

Moreover, if ρ has compact image and E has order continuous norm or $E = \ell^\infty$, then every invariant and band irreducible band is finite dimensional, and so $\alpha = \infty$.

Examples of normalized symmetric Banach sequence spaces with order continuous norm are the classical sequence spaces c_0 and ℓ^p for $1 \leq p < \infty$.

In general, one cannot expect a direct sum decomposition into band irreducibles as in the above theorem for positive representations of compact groups in arbitrary

Banach lattices. An example is the representation of the trivial group in $L^p([0,1])$, $1 \leq p \leq \infty$, where there are no nonzero band irreducible subrepresentations at all. In order to obtain some kind of decomposition in other spaces, some new ideas are needed. A result in this direction is [21], in which composition series of ordered structures are examined. Another result is [23], where positive representations of L^p-spaces associated with Polish transformation groups are considered. In that paper it is shown that for such representations, a decomposition into band irreducibles exists, in terms of Banach bundles, which is at least in spirit close to the direct integral of Hilbert spaces used in Theorem 1.1.

It is clear that there is still a lot of work to be done concerning decomposition theorems for positive representations, but the first steps have now been taken.

Chapter 2

Crossed products of Banach algebras

This chapter is to appear in Dissertationes Mathematicae as: M. de Jeu, S. Dirksen and M. Wortel, "Crossed products of Banach algebras. I.". It is available as arXiv:1104.5151.

Abstract. We construct a crossed product Banach algebra from a Banach algebra dynamical system (A, G, α) and a given uniformly bounded class $\mathcal{R}$ of continuous covariant Banach space representations of that system. If A has a bounded left approximate identity, and $\mathcal{R}$ consists of non-degenerate continuous covariant representations only, then the non-degenerate bounded representations of the crossed product are in bijection with the non-degenerate $\mathcal{R}$-continuous covariant representations of the system. This bijection, which is the main result of the paper, is also established for involutive Banach algebra dynamical systems and then yields the well-known representation theoretical correspondence for the crossed product C^*-algebra as commonly associated with a C^*-algebra dynamical system as a special case. Taking the algebra A to be the base field, the crossed product construction provides, for a given non-empty class of Banach spaces, a Banach algebra with a relatively simple structure and with the property that its non-degenerate contractive representations in the spaces from that class are in bijection with the isometric strongly continuous representations of G in those spaces. This generalizes the notion of a group C^*-algebra, and may likewise be used to translate issues concerning group representations in a class of Banach spaces to the context of a Banach algebra, simpler than $L^1(G)$, where more functional analytic structure is present.

2.1 Introduction

The theory of crossed products of C^*-algebras started with the papers by Turumaru [49] from 1958 and Zeller-Meier from 1968 [54]. Since then the theory has been extended extensively, as is attested by the material in Pedersen's classic [35] and, more recently, in Williams' monograph [51]. Starting with a C^*-dynamical system (A, G, α), where A is a C^*-algebra, G is a locally compact group, and α a strongly continuous action of G on A as involutive automorphisms, the crossed product construction yields a C^*-algebra $A \rtimes_\alpha G$ which is built from these data. Thus the crossed product construction provides a means to construct examples of C^*-algebras from, in a sense, more elementary ingredients. One of the basic facts for a crossed product C^*-algebra $A \rtimes_\alpha G$ is that the non-degenerate involutive representations of this algebra on Hilbert spaces are in one-to-one correspondence with the non-degenerate involutive continuous covariant representations of (A, G, α), i.e., with the pairs (π, U), where π is a non-degenerate involutive representation of A on a Hilbert space, and U is a unitary strongly continuous representations of G on the same space, such that the covariance condition $\pi(\alpha_s(a)) = U_s \pi(a) U_s^{-1}$ is satisfied, for $a \in A$, and $s \in G$.

This paper contains the basics for the natural generalization of this construction to the general Banach algebra setting. Starting with a Banach algebra dynamical system (A, G, α), where A is a Banach algebra, G is a locally compact group, and α a strongly continuous action of G on A as not necessarily isometric automorphisms, we want to build a Banach algebra of crossed product type from these data. Moreover, we want the outcome to be such that (suitable) non-degenerate continuous covariant representations of (A, G, α) are in bijection with (suitable) non-degenerate bounded representations of this crossed product Banach algebra. Later in this introduction, more will be said about how to construct such an algebra, and how the construction can be tuned to accommodate a class $\mathcal{R}$ of non-degenerate continuous covariant representations relevant for the case at hand. It will then also become clear what being "suitable" means in this context. For the moment, let us oversimplify a bit and, neglecting the precise hypotheses, state that such an algebra can indeed be constructed. The precise statement is Theorem 2.8.1, which we will discuss below.

Before continuing the discussion of crossed products of Banach algebra as such, however, let us mention our motivation to start investigating these objects, and sketch perspectives for possible future applications of our results. Firstly, just as in the case of a crossed product C^*-algebra, it simply seems natural as such to have a means available to construct Banach algebras from more elementary building blocks. Secondly, there are possible applications of these algebras in the theory of Banach representations of locally compact groups. We presently see two of these, which we will now discuss.

Starting with the first one, we recall that, as a special case of the correspondence for crossed product C^*-algebras mentioned above, the unitary strongly continuous representations of a locally compact group G in Hilbert spaces are in bijection with the non-degenerate involutive representations of the group C^*-algebra $C^*(G) = \mathbb{C} \rtimes_{\mathrm{triv}} G$ in Hilbert spaces. It is by this fact that questions concerning, e.g., the existence of sufficiently many irreducible unitary strongly continuous representations of G to separate its points, and, notably, the decomposition of an arbitrary unitary strongly continuous representation of G into irreducible ones, can be translated to C^*-algebras and solved in that context [10]. For Banach space representations of G, the theory is considerably less well developed. To our knowledge, the only available general decomposition theorem, comparable to those in a unitary context, is Shiga's [46], stating that a strongly continuous representation of a compact group in an arbitrary Banach space decomposes in a Peter-Weyl–fashion, analogous to that for a unitary strongly continuous representation in a Hilbert space. With the results of the present paper, it is possible to construct Banach algebras which, just as the group C^*-algebra, are "tuned" to the situation. Our main results in this direction are Theorem 2.9.7 and Theorem 2.9.8. The latter, for example, yields, for any non-empty class $\mathcal{X}$ of Banach spaces, a Banach algebra $\mathcal{B}_{\mathcal{X}}(G)$, such that the non-degenerate contractive representations of $\mathcal{B}_{\mathcal{X}}(G)$ in spaces from $\mathcal{X}$ are in bijection with the isometric strongly continuous representations of G in these spaces. This algebra $\mathcal{B}_{\mathcal{X}}(G)$ could be called the group Banach algebra of G associated with $\mathcal{X}$, and, as will become clear in Section 2.9.2, only the isometric strongly continuous representations of G in the spaces from $\mathcal{X}$ are used in its construction. The analogy with the group C^*-algebra $C^*(G)$, which is in fact a special case, is clear. Just as is known to be the case with $C^*(G)$, one may hope that, for certain classes $\mathcal{X}$ of sufficiently well-behaved spaces, the study of $\mathcal{B}_{\mathcal{X}}(G)$ will shed light on the theory of isometric strongly continuous representations of G in the spaces from $\mathcal{X}$. For comparison, we recall the well-known fact [20, Assertion VI.1.32], [24, Proposition 2.1] that the non-degenerate bounded representations of $L^1(G)$ in a Banach space are in bijection with the uniformly bounded strongly continuous representations of G in that Banach space. So, certainly, there is already a Banach algebra available to translate questions concerning group representations to, but the point is that it is very complicated, simply because $L^1(G)$ apparently carries the information of all such representations of G in *all* Banach spaces. One may hope that, for certain choices of $\mathcal{X}$, an algebra such as $\mathcal{B}_{\mathcal{X}}(G)$, the construction of which uses no more data than evidently necessary, has a sufficiently simpler structure than $L^1(G)$ to admit the development of a reasonable representation theory, and hence for the isometric strongly continuous representations of G in these spaces, thus paralleling the case where $\mathcal{X}$ consists of all Hilbert spaces and $B_{\mathcal{X}}(G) = C^*(G)$. Aside, let us mention that $L^1(G)$ is, in fact, isometrically isomorphic to a crossed product $(\mathbb{F} \rtimes_{\mathrm{triv}} G)^{\mathcal{R}}$ as in the present paper, if one chooses the class $\mathcal{R}$—to be discussed below—appropriately. In that case, it is possible to infer the aforementioned bijection between the non-degenerate bounded representations of $L^1(G)$ and the uniformly bounded strongly continuous representations of G from Theorem 2.8.1, due to the fact that these representations of G can then be seen to correspond to the $\mathcal{R}$-continuous—also to be discussed below—representations of $(\mathbb{F}, G, \mathrm{triv})$. In view of the further increase in length of the present paper that would be a consequence of the inclusion of these and further related results, we have decided to postpone these to the sequel [22], including only some preparations for this at the end of Section 2.9.1.

The second possible application in group representation theory lies in the relation between imprimitivity theorems and Morita equivalence. Whereas the construction of the group Banach algebras $B_{\mathcal{X}}(G)$ and establishing their basic properties could be done in a paper quite a bit shorter than the present one, the general crossed product construction and ensuing results are indeed needed for this second perspective.

Starting with the involutive context, we recall that Mackey's now classical result [30] asserts that a unitary strongly continuous representation U of a separable locally compact group G is unitarily equivalent to an induced unitary strongly continuous representation of a closed subgroup H, precisely when there exists a system of imprimitivity $(G/H, U, P)$ based on the G-space G/H. The separability condition of G is actually not necessary, as shown by Loomis [27] and Blattner [5], and for general G Mackey's imprimitivity theorem can be reformulated as ([40, Theorem 7.18]): U is unitarily equivalent to such an induced representation precisely when there exists a non-degenerate involutive representation π of $C_0(G/H)$ in the same Hilbert space, such that (π, U) is a covariant representation of the C^*-dynamical system $(C_0(G/H), G, \mathrm{lt})$, where G acts as left translations on $C_0(G/H)$. Using the standard correspondence for crossed products of C^*-algebras, one thus sees that, up to unitary equivalence, such U are precisely the unitary parts of the non-degenerate involutive continuous covariant representations of the C^*-dynamical system $(C_0(G/H), G, \mathrm{lt})$ corresponding to the non-degenerate involutive representations of the crossed product C^*-algebra $C_0(G/H) \rtimes_{\mathrm{lt}} G$. Rieffel's theory of induction for C^*-algebras [37], [40] and Morita-equivalence [38], [41] allows us to follow another approach to Mackey's theorem, as was in fact done in [40], by proving that $C_0(G/H) \rtimes_{\mathrm{lt}} G$ and $C^*(H)$ are (strongly) Morita equivalent as a starting point. This implies that these C^*-algebras have equivalent categories of non-degenerate involutive representations, and working out this correspondence then yields Mackey's imprimitivity theorem. For more detailed information we refer to [40], [38], and [41], as well as (also including significant further developments) to [15], [31], [36], [12], [51] and [14], the latter also for Banach *-algebras and Banach *-algebraic bundles.

The Morita theorems in a purely algebraic context are actually more symmetric than the analogous ones in Rieffel's work. We formulate part of the results for algebras over a field k (cf. [13, Theorem 12.12]): If A and B are unital k-algebras, then the categories of left A-modules and left B-modules are k-linearly equivalent precisely when there exist bimodules ${}_AP_B$ and ${}_BQ_A$, such that $P \otimes_B Q \simeq A$ as A-A-bimodules, and $Q \otimes_A P \simeq B$ as B-B-bimodules. From the existence of such bimodules it follows easily that the categories are equivalent, since equivalence are manifestly given by $M \mapsto Q \otimes_A M$, for a left A-module M, and by $N \mapsto P \otimes_B N$, or a left B-module N. The non-trivial statement is that the converse is equally true. In Rieffel's analytical context, the role of the bimodules P and Q for C^*-algebras A and B is taken over by so-called imprimitivity bimodules, sometimes also called equivalence bimodules. These are A-B-Hilbert C^*-modules ([36, Definition 3.1]), and the existence of such imprimitivity bimodules (actually, exploiting duality, only one is needed, see [36, p. 49]) implies that the categories of non-degenerate involutive representations of these C^*-algebras are equivalent [36, Theorem 3.29]. In contrast with the algebraic context, the converse is not generally true (see [36, Remark 3.15 and Hooptedoodle 3.30]). This has led to the distinction between strong Morita equivalence (in the sense of existing imprimitivity bimodules) and weak Morita equivalence (in the sense of equivalent categories of non-degenerate involutive representations) of C^*-algebras. The work of Blecher [7], generalizing earlier results of Beer [4], shows how to remedy this: if one enlarges the categories, taking them to consist of all left A-operator modules as objects and completely bounded A-linear maps as morphisms, and similarly for B, then symmetry is restored as in the algebraic case: the equivalence of these larger categories is then equivalent with the existence of an imprimitivity bimodule, i.e., with strong Morita equivalence of the C^*-algebras in the sense of Rieffel. As a further step, strong Morita equivalence was developed for operator algebras (i.e., norm-closed subalgebras of $B(H)$, for some Hilbert space H) by Blecher, Muhly and Paulsen in [8]. Restoring symmetry again, Blecher proved in [6] that, for operator algebras with a contractive approximate identity, strong Morita equivalence is equivalent to their categories of operator modules being equivalent via completely contractive functors.

A part of the well-developed theory in a Hilbert space context as mentioned above has a parallel for Banach algebras and representations in Banach spaces, but, as far as we are aware, the body of knowledge is much smaller than for Hilbert spaces.[1] Induction of representations of locally compact groups and Banach algebras in Banach spaces has been investigated by Rieffel in [39], from the categorical viewpoint that, as a functor, induction is, or ought to be, an adjoint of the restriction functor. In [18], Grønbæk studies Morita equivalence for Banach algebras in a context of Banach space representations, and a Morita-type theorem [16, Corollary 3.4] is established for Banach algebras with bounded two-sided approximate identities: such Banach algebras A and B have equivalent categories of non-degenerate left Banach modules precisely when there exist non-degenerate Banach bimodules ${}_AP_B$ and ${}_BQ_A$, such that $P\widehat{\otimes}_B Q \simeq A$ as A-A-bimodules, and $Q\widehat{\otimes}_A P \simeq B$ as B-B-bimodules. In subsequent work [17], this result is generalized to self-induced Banach algebras, and this generalization yields an imprimitivity theorem [18, Theorem IV.9] in a form quite similar to Mackey's theorem as formulated by Rieffel [40, Theorem 7.18] (i.e., with a $C_0(G/H)$-action instead of a projection valued measure), with a continuity condition on the action of $C_0(G/H)$. The approach of this imprimitivity theorem, via Morita equivalence of Banach algebras, is therefore analogous to Rieffel's work, and here again algebras which are called crossed products make their appearance [18, Definition IV.1]. Given the results in the present paper, it is natural to ask whether this imprimitivity theorem (or a variation of it) can also be derived from a surmised Morita equivalence of the crossed product Banach algebra $(C_0(G/H) \rtimes_{\mathrm{lt}} G)^{\mathcal{R}}$ and a group Banach algebra $B_{\mathcal{X}}(H)$ as in the present paper (for suitable $\mathcal{R}$ and $\mathcal{X}$), and what the relation is between the algebras in [18, Definition IV.1], also called crossed products, and the crossed product Banach algebras in the present paper. We expect to investigate this in the future, also taking the work of De Pagter and Ricker [34] into account. In that paper, it is shown that, for certain bounded Banach space representations (including all bounded representations in reflexive spaces[2]) of $C(K)$, where K is a compact Hausdorff space, there is always an underlying projection valued measure. In such cases, if G/H is compact (and it is perhaps not overly optimistic to expect that the results in [34] can be generalized to the locally

[1] As an illustration: as far as we know, for groups, [29] is currently the only available book on Banach space representations.

[2] In fact: including all bounded representations in spaces not containing a copy of c_0, see [34, Corollary 2.16].

compact case, so as to include representations of $C_0(G/H)$ for non-compact G/H), an imprimitivity theorem for Banach space representations of groups can be derived in Mackey's original form in terms of systems of imprimitivity. If all this comes to be, then this would be a satisfactory parallel—for suitable Banach spaces—with the Hilbert space context, both in the spirit of Rieffel's strong Morita equivalence of $C_0(G/H) \rtimes_{\mathrm{lt}} G$ and $C^*(H)$ as a means to obtain an imprimitivity theorem, and of Mackey's systems of imprimitivity as a means to formulate such a theorem. We hope to be able to report on this in due time.

We will now outline the mathematical structure of the paper. Although the crossed product of a general Banach algebra is more involved than its C^*-algebra counterpart, the reader may still notice the evident influence of [51] on the present paper. We start by explaining how to construct the crossed product. Given a Banach algebra dynamical system (A, G, α) (Definition 2.2.10), and a non-empty class $\mathcal{R}$ of continuous covariant representations (Definition 2.2.12), we want to introduce an algebra seminorm $\sigma^{\mathcal{R}}$ on the twisted convolution algebra $C_c(G, A)$ by defining

$$\sigma^{\mathcal{R}}(f) = \sup_{(\pi,U)\in\mathcal{R}} \left\| \int_G \pi(f(s))U_s \, ds \right\| \quad (f \in C_c(G, A)).$$

For a C^*-dynamical system, if one lets $\mathcal{R}$ consist of all pairs (π, U) where π is involutive and non-degenerate, and U is unitary and strongly continuous, this supremum is evidently finite, and $\sigma^{\mathcal{R}}$ is even a norm. For a general Banach algebra dynamical system, neither need be the case. This therefore leads us, first of all, to introduce the notion of a uniformly bounded (Definition 2.3.1) class of covariant representations, in order to ensure the finiteness of $\sigma^{\mathcal{R}}$. The resulting crossed product Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ is then, by definition, the completion of $C_c(G, A)/\ker(\sigma^{\mathcal{R}})$ in the algebra norm induced by $\sigma^{\mathcal{R}}$ on this quotient. Thus, as a second difference with the construction of the crossed product C^*-algebra associated with a C^*-dynamical system, a non-trivial quotient map is inherent in the construction.

While the construction is thus easily enough explained, the representation theory, to which we now turn, is more involved. Suppose that (π, U) is a continuous covariant representation of (A, G, α), and that there exists $C \geq 0$, such that $\left\| \int_G \pi(f(s))U_s \, ds \right\| \leq C\sigma^{\mathcal{R}}(f)$, for all $f \in C_c(G, A)$. In that case, we say that (π, U) is $\mathcal{R}$-continuous, and it is clear that there is an associated bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$, denoted by $(\pi \rtimes U)^{\mathcal{R}}$. Certainly all elements of $\mathcal{R}$ are $\mathcal{R}$-continuous, yielding even contractive representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$, but, as it turns out, there may be more. Likewise, $(A \rtimes_\alpha G)^{\mathcal{R}}$ may have non-contractive bounded representations. This contrasts the analogous involutive context for the crossed product C^*-algebra associated with a C^*-dynamical system. The natural question is, then, what the precise relation is between the $\mathcal{R}$-continuous covariant representations of (A, G, α) and the bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$. The answer turns out to be quite simple: if A has a bounded left approximate identity, and if $\mathcal{R}$ consists of non-degenerate (Definition 2.2.12) continuous covariant representations only, then the map $(\pi, U) \mapsto (\pi \rtimes U)^{\mathcal{R}}$ is a bijection between the non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) and the non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$. This is the main content of Theorem 2.8.1.

Establishing this, however, is less simple. The first main step to be taken is to construct any representations of the group and the algebra at all from a given (non-degenerate) bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$. In case of a crossed product C^*-algebra and involutive representations in Hilbert spaces, there is a convenient way to proceed [51]. One starts by viewing this crossed product as an ideal of its double centralizer algebra. If the involutive representation T of the crossed product C^*-algebra is non-degenerate, then it can be extended to an involutive representation of the double centralizer algebra. Subsequently, it can be composed with existing homomorphisms of group and algebra into this double centralizer algebra, thus yielding a pair (π, U) of representations. These can then be shown to have the desired continuity, involutive and covariance properties and, moreover, the corresponding non-degenerate involutive representation of the crossed product C^*-algebra turns out to be T again. For Banach algebra dynamical systems we want to use a similar circle of ideas, but here the situation is more involved. To start with, it is not necessarily true that a Banach algebra $\mathcal{A}$ can be mapped injectively into its double centralizer algebra $\mathcal{M}(\mathcal{A})$, or that a non-degenerate representation of $\mathcal{A}$ necessarily comes with an associated representation of the double centralizer algebra, compatible with the natural homomorphism from $\mathcal{A}$ into $\mathcal{M}(\mathcal{A})$. This question motivated the research leading to [9] as a preparation for the present paper, and, as it turns out, such results can be obtained. For example, if the algebra $\mathcal{A}$ has a bounded left approximate identity, and a non-degenerate bounded representation of $\mathcal{A}$ is given, then there is an associated bounded representation of the left centralizer algebra $\mathcal{M}_l(\mathcal{A})$ which is compatible with the natural homomorphism from $\mathcal{A}$ into $\mathcal{M}_l(\mathcal{A})$, with similar results for right and double centralizer algebras.[3] If we want to apply this in our situation, then we need to show that $(A \rtimes_\alpha G)^{\mathcal{R}}$ has a bounded left approximate identity. For crossed product C^*-algebras, this is of course automatic, but in the present case it is not. Thus it becomes necessary to establish this independently, and indeed $(A \rtimes_\alpha G)^{\mathcal{R}}$ has a bounded approximate left identity if A has one, with similar right and two-sided results. As an extra complication, since the representations of A under consideration are now not necessarily contractive anymore, and the group need not act isometrically, it becomes necessary, with the future applications in Section 2.9 in mind, to keep track of the available upper bounds for the various maps as they are constructed during the process. For this, in turn, one needs an explicit upper bound for bounded left and right approximate identities in $(A \rtimes_\alpha G)^{\mathcal{R}}$. It is for these reasons that Section 2.4 on approximate identities in $(A \rtimes_\alpha G)^{\mathcal{R}}$ and their bounds, which is superfluous for crossed product C^*-algebras, is a key technical interlude in the present paper.

After that, once we know that $(A \rtimes_\alpha G)^{\mathcal{R}}$ has a left bounded approximate identity, we can let the left centralizer algebra $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ take over the role that the double centralizer algebra has for crossed product C^*-algebras. Given a non-degenerate bounded representation T of $(A \rtimes_\alpha G)^{\mathcal{R}}$, we can now find a compatible

[3] Theorem 2.6.1 contains a summary of what is needed in the present paper.

non-degenerate bounded representation of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$, and on composing this with existing homomorphisms of the algebra and the group into $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$, we obtain a pair (π, U) of representations. The continuity and covariance properties are easily established, as is the non-degeneracy of π, but as compared to the situation for crossed product C^*-algebras, a complication arises again. Indeed, since in that case $\mathcal{R}$ consists of all non-degenerate involutive covariant representations of (A, G, α) in Hilbert spaces, and an involutive T yields and involutive π and unitary U, the pair (π, U) is automatically in $\mathcal{R}$, and is therefore certainly $\mathcal{R}$-continuous. For Banach algebra dynamical systems this need not be the case, and the norm estimates in our bookkeeping, although useful in Section 2.9, provide no rescue: one needs an independent proof to show that (π, U) as obtained from T is $\mathcal{R}$-continuous. Once this has been done, it is not overly complicated anymore to show that the associated bounded representation $(\pi \rtimes U)^{\mathcal{R}}$ of $(A \rtimes_\alpha G)^{\mathcal{R}}$ (which can then be defined) is T again. By keeping track of invariant closed subspaces and bounded intertwining operators during the process, and also considering the involutive context at little extra cost, the basic correspondence in Theorem 2.8.1 has then finally been established.

With this in place, and also the norm estimates from our bookkeeping available, it is easy so give applications in special situations. This is done in the final section, where we formulate, amongst others, the results for group Banach algebras $B_{\mathcal{X}}(G)$ already mentioned above. We then also see that the basic representation theoretical correspondence for "the" C^*-crossed product as commonly associated with a C^*-dynamical system is an instance of a more general correspondence (Theorem 2.9.3), valid for C^*-algebras of crossed product type associated with an involutive (Definition 2.2.10) Banach algebra dynamical systems (A, G, α), provided that, for all $\varepsilon > 0$, A has a $(1+\varepsilon)$-bounded approximate left identity.

This paper is organized as follows.

In Section 2.2 we establish the necessary basic terminology and collect some preparatory technical results for the sequel. Some of these can perhaps be considered to be folklore, but we have attempted to make the paper reasonably self-contained, especially since the basics for a general Banach algebra and Banach space situation are akin, but not identical, to those for C^*-algebras and Hilbert spaces, and less well-known. At the expense of a little extra verbosity, we have also attempted to be as precise as possible, throughout the paper, by including the usual conditions, such as (strong) continuity or (in the case of algebras) non-degeneracy of representations, only when they are needed and then always formulating them explicitly, thus eliminating the need to browse back and try to find which (if any) convention applies to the result at hand. There are no such conventions in the paper. It would have been convenient to assume from the very start that, e.g., all representations are (strongly) continuous and (in case of algebras) non-degenerate, but it seemed counterproductive to do so.

Section 2.3 contains the construction of the crossed product and its basic properties. The ingredients are a given Banach algebra dynamical system (A, G, α) and a uniformly bounded class $\mathcal{R}$ of continuous covariant representations thereof.

Section 2.4 contains the existence results and bounds for approximate identities in $(A \rtimes_\alpha G)^{\mathcal{R}}$. As explained above, this is a key issue which need not be addressed in the case of crossed product C^*-algebras.

Section 2.5 is concerned with the easiest part of the representation theory as considered in this paper: the passage from $\mathcal{R}$-continuous covariant representations of (A, G, α) to bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$. We have included results about preservation of invariant closed subspaces, bounded intertwining operators and non-degeneracy. In this section, two homomorphisms i_A and i_G of, respectively, A and G into End $(C_c(G, A))$ make their appearance, which will later yield homomorphisms $i_A^{\mathcal{R}}$ and $i_G^{\mathcal{R}}$ into the left centralizer algebra $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$, as needed to construct a covariant representation of (A, G, α) from a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$. With the involutive case in mind, anti-homomorphism j_A and j_G into End $(C_c(G, A))$ are also considered.

Section 2.6 on centralizer algebras starts with a review of part of the results from [9], and then, after establishing a separation property to be used later (Proposition 2.6.2), continues with the study of more or less canonical (anti-)homomorphisms of A and G into the left, right or double centralizer algebra of $(A \rtimes_\alpha G)^{\mathcal{R}}$. These (anti-)homomorphisms, such as $i_A^{\mathcal{R}}$ and $i_G^{\mathcal{R}}$ already alluded to above are based on the (anti-)homomorphisms from Section 2.5.

Section 2.7 contains the most involved part of the representation theory: the passage from non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ to non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α). At this point, if A has a bounded left approximate identity, then Sections 2.4 and 2.6 provide the necessary ingredients. If T is a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$, then there is a compatible non-degenerate bounded representation $\overline{T}$ of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$, and one thus obtains a representation $\overline{T} \circ i_A^{\mathcal{R}}$ of A and a representation $\overline{T} \circ i_G^{\mathcal{R}}$ of G. The main hurdle, namely to construct any representations of A and G at all from T, has thus been taken, but still some work needs to be done to take care of the remaining details.

Section 2.8 contains, finally, the bijection between the non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) and the non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$, valid if A has a bounded left approximate identity and $\mathcal{R}$ consists of non-degenerate continuous covariant representations only. Obtaining this Theorem 2.8.1 is simply a matter of putting the pieces together. Results about preservation of invariant closed subspaces and bounded intertwining operators are also included, as is a specialization to the involutive case. For convenience, we have also included in this section some relevant explicit expressions and norm estimates as they follow from the previous material.

In Section 2.9 the basic correspondence from Theorem 2.8.1 is applied to various situations, including the cases of a trivial algebra and of a trivial group. Whereas an application of this theorem to the case of a trivial algebra does lead to non-trivial results about group Banach algebras, as discussed earlier in this Introduction, it does not give optimal results for a trivial group. In that case, the machinery of the present paper is, in fact, largely superfluous, but for the sake of completeness we have nevertheless included a brief discussion of that case and a formulation of the (elementary) optimal results.

Reading guide. In the discussion above it may have become evident that, whereas the construction of a Banach algebra crossed product requires modifications of the crossed product C^*-algebra construction which are fairly natural and easily implemented, establishing the desired correspondence at the level of (covariant) representations is more involved than for crossed product C^*-algebras. As evidence of this may serve the fact that Theorem 2.8.1 can, without too much exaggeration, be regarded as the summary of most material preceding it, including some results from [9]. To facilitate the reader who is mainly interested in this correspondence as such, and in its applications in Section 2.9, we have included (references to) the relevant definitions in Sections 2.8 and 2.9. We hope that, with some browsing back, these two sections, together with this Introduction, thus suffice to convey how $(A \rtimes_\alpha G)^{\mathcal{R}}$ is constructed and what its main properties and special cases are.

Perspectives. According to its preface, [51] can only cover part of what is currently known about crossed products of C^*-algebras in one volume. Although the theory of crossed products of Banach algebras is, naturally, not nearly as well developed as for C^*-algebras, it is still true that more can be said than we felt could reasonably be included in one research paper. Therefore, in [22] we will continue the study of these algebras. We plan to include (at least) a characterization of $(A \rtimes_\alpha G)^{\mathcal{R}}$ by a universal property in the spirit of [51, Theorem 2.61], as well as a detailed discussion of L^1-algebras. As mentioned above, $L^1(G)$ is isometrically isomorphic to a crossed product as constructed in the present paper, and the well-known link between its representation theory and that for G follows from our present results. We will include this, as a special case of similar results for $L^1(G, A)$ with twisted convolution. Also, we will then consider natural variations on the bijection theme: suppose that one has, say, a uniformly bounded class $\mathcal{R}$ of pairs (π, U), where π is a non-degenerate continuous anti-representation of A, U is a strongly continuous anti-representation of G, and the pair (π, U) is anti-covariant, is it then possible to find an algebra of crossed product type, the non-degenerate bounded anti-representations of which correspond bijectively to the $\mathcal{R}$-continuous pairs (π, U) with the properties as just mentioned? It is not too difficult to relate these questions to the results in the present paper, albeit sometimes for a closely related alternative Banach algebra dynamical system, and it seems quite natural to consider this matter, since examples of such $\mathcal{R}$ are easy to provide. Once this has been done, we will also be able to infer the basic relation [24, Proposition 2.1] between $L^1(G)$-*bi*modules and G-*bi*modules from the results in the present paper. As mentioned above, we also plan to consider Morita equivalence and imprimitivity theorems, but that may have to wait until another time. The same holds for crossed products of Banach algebras in the context of positive representations on Banach lattices.[4]

[4]As a preparation, positivity issues have already been taken into account in [9].

2.2 Preliminaries

In this section we introduce the basic definitions and notations, and establish some preliminary results. We start with a few general notions.

If G is a group, then e will be its identity element. If G is a locally compact group, then we fix a left Haar measure μ on G, and denote integration of a function ψ with respect to this Haar measure by $\int_G \psi(s)\, ds$. We let $\Delta : G \to (0, \infty)$ denote the modular function, so for $f \in C_c(G)$ and $r \in G$ we have ([51, Lemma 1.61, Lemma 1.67])

$$\Delta(r) \int_G f(sr)\, ds = \int_G f(s)\, ds, \quad \int_G \Delta(s^{-1}) f(s^{-1})\, ds = \int_G f(s)\, ds.$$

If X is a normed space, we denote by $B(X)$ the normed algebra of bounded operators on X. We let $\mathrm{Inv}(X)$ denote the group of invertible elements of $B(X)$. If A is a normed algebra, we write $\mathrm{Aut}^+(A)$ for its group of bounded automorphisms.

A neighbourhood of a point in a topological space is a set with that point as interior point. It is not necessarily open.

Throughout this paper, the scalar field can be either the real or the complex numbers.

2.2.1 Group representations

Definition 2.2.1. A *representation* U of a group G on a normed space X is a group homomorphism $U : G \to \mathrm{Inv}(X)$.

Note that there is no continuity assumption, which is actually quite convenient during proofs. For typographical reasons, we will write U_s rather than $U(s)$, for $s \in G$.

Lemma 2.2.2. *Let X be a non-zero Banach space and U be a strongly continuous representation of a topological group G on X. Then for every compact set $K \subset G$ there exist a constant $M_K > 0$ such that, for all $r \in K$,*

$$\frac{1}{M_K} \leq \|U_r\| \leq M_K.$$

Proof. For fixed $x \in X$, the map $r \mapsto U_r x$ is continuous, so the set $\{U_r x : r \in K\}$ is compact and hence bounded. By the Banach-Steinhaus Theorem, there exists $M'_K > 0$ such that $\|U_r\| \leq M'_K$ for all $r \in K$. Since, for $r \in K$,

$$1 = \|\mathrm{id}_X\| \leq \|U_{r^{-1}}\| \, \|U_r\| \leq M'_{K^{-1}} \, \|U_r\| \,,$$

$M_K = \max(M'_K, M'_{K^{-1}})$ is as required. □

If U is a strongly continuous representation of a topological group G on a Banach space X, then the natural map from $G \times X$ to X is separately continuous. Actually, it is automatically jointly continuous, according to the next result.

Proposition 2.2.3. *Let U be a strongly continuous representation of the locally compact group G on the Banach space X. Then the map $(r, x) \to U_r x$ from $G \times X$ to X is continuous.*

Proof. We may assume that X is non-zero. Fix $(r_0, x_0) \in G \times X$ and let $\varepsilon > 0$. There exists a neighbourhood V of r_0 such that, for all $r \in V$, $\|U_r x_0 - U_{r_0} x_0\| < \varepsilon/2$. We may assume that V is compact, and then Lemma 2.2.2 yields an $M_V > 0$ such that $\|U_r\| \leq M_V$ for all $r \in V$. Therefore, if $r \in V$ and $\|x - x_0\| < \varepsilon/(2M_V)$, we have

$$\begin{aligned}\|U_r x - U_{r_0} x_0\| &\leq \|U_r x - U_r x_0\| + \|U_r x_0 - U_{r_0} x_0\| \\ &< M_V \cdot \frac{\varepsilon}{2M_V} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

□

Corollary 2.2.5 below, and notably its second statement, will be used repeatedly when showing that a representation of a locally compact group is strongly continuous. The following lemma is a preparation.

Lemma 2.2.4. *Let G and H be two groups with a topology such that right multiplication is continuous in both groups, or such that left multiplication is continuous in both groups. Let $U : G \to H$ be a homomorphism. Then U is continuous if and only if it is continuous at e.*

Proof. Assume that right multiplication is continuous in both groups. Let U be a homomorphism which is continuous at e and let $(r_i) \in G$ be a net converging to $r \in G$. Then $r_i r^{-1} \to e$ by the continuity of right multiplication by r^{-1} in G, and so

$$U_{r_i} = U_{r_i r^{-1}} U_r \to U_r,$$

where the continuity of right multiplication by U_r in H is used in the last step. The case of continuous left multiplication is proved similarly, writing $U_{r_i} = U_r U_{r^{-1} r_i}$. □

Corollary 2.2.5. *Let G be a group with a topology such that right or left multiplication is continuous. Let X be a Banach space and suppose $U : G \to \mathrm{Inv}(X)$ is a representation of G on X. Then U is a strongly continuous representation if and only U is strongly continuous at e. If U is uniformly bounded on some neighbourhood of e, and $Y \subset X$ is a dense subset of X, then U is a strongly continuous representation if and only if $r \mapsto U_r y$ is continuous at e for all $y \in Y$.*

Proof. The first part follows from Lemma 2.2.4 and the fact that multiplication in $B(X)$ equipped with the strong operator topology is separately continuous. The second part is an easy consequence of the first. □

If X is a Hilbert space, then the $*$-*strong operator topology* is the topology on $B(X)$ generated by the seminorms $T \mapsto \|Tx\| + \|T^* x\|$, with $x \in X$. A net

(T_i) converges $*$-strongly to T if and only if both $T_i \to T$ strongly and $T_i^* \to T^*$ strongly. This topology is stronger than the strong operator topology and weaker than the norm topology, and multiplication is continuous in this topology on uniformly bounded subsets of $B(X)$.

Remark 2.2.6. If U is a unitary representation, then the decomposition of $r \mapsto U_r^*$ as $r \mapsto r^{-1} \mapsto U_r^{-1} = U_r^*$ shows that U is strongly continuous if and only if U is $*$-strongly continuous.

2.2.2 Algebra representations

Definition 2.2.7. A *representation* π of an algebra A on a normed space X is an algebra homomorphism $\pi : A \to B(X)$. The representation π is non-degenerate if $\pi(A) \cdot X := \operatorname{span}\{\pi(a)x : a \in A, x \in X\}$ is dense in X.

Note that it is not required that π is unital if A has a unit element, nor that π is (norm) bounded if A is a normed algebra.

Remark 2.2.8. If A is a normed algebra with a bounded left approximate identity (u_i), and π is a bounded representation of A on the Banach space X, then it is easy to verify that π is non-degenerate if and only if $\pi(u_i) \to \mathrm{id}_X$ in the strong operator topology.

The following result, which will be used in the context of covariant representations, follows readily using Remark 2.2.8.

Lemma 2.2.9. *Let A be a normed algebra with a bounded approximate left identity, and let π be a bounded representation of A on a Banach space X.*

(i) If π is non-degenerate and $Z \subset X$ is an invariant subspace, then the restricted representation of A to Z is non-degenerate.

(ii) There is a largest invariant subspace such that the restricted representation of A to it is non-degenerate. This subspace is closed. In fact, it is $\overline{\pi(A) \cdot X}$.

2.2.3 Banach algebra dynamical systems and covariant representations

We continue by defining the notion of a dynamical system in our setting.

Definition 2.2.10. A *normed (resp. Banach) algebra dynamical system* is a triple (A, G, α), where A is a normed (resp. Banach) algebra[5], G is a locally compact Hausdorff group, and $\alpha : G \to \mathrm{Aut}^+(A)$ is a strongly continuous representation of G on A. The system is called involutive when the scalar field is $\mathbb{C}$, A has a bounded involution and α_s is involutive for all $s \in G$.

[5] If A is an algebra, then we do not assume that it is unital, nor that, if it is a unital normed algebra, the identity element has norm 1.

From Proposition 2.2.3 we see that, for a Banach algebra dynamical system (A, G, α), the canonical map $(s, a) \to \alpha_s(a)$ is continuous from $G \times A$ to A. This fact has as important consequence that a number of integrands in the sequel are *continuous* vector valued functions on G, and we mention one of these explicitly for future reference.

Lemma 2.2.11. *Let (A, G, α) be a Banach algebra dynamical system, let $s \in G$ and $f \in C(G, A)$. Then the map $r \mapsto \alpha_r\left(f(r^{-1}s)\right)$ from G to A is continuous.*

Indeed, this map is the composition of the maps $r \mapsto (r, f(r^{-1}s))$ from G to $G \times A$ and the canonical map from $G \times A$ to A.

Next we define our main objects of interest, the covariant representations.

Definition 2.2.12. Let (A, G, α) be a normed algebra dynamical system, and let X be a normed space. Then a *covariant representation* of (A, G, α) on X is a pair (π, U), where π is a representation of A on X and U is a representation of G on X, such that for all $a \in A$ and $s \in G$,

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^{-1}.$$

The covariant representation (π, U) is called continuous if π is norm bounded and U is strongly continuous, and it is called non-degenerate if π is a non-degenerate representation of A.

If (A, G, α) is a normed algebra dynamical system, then the covariant representation (π, U) of (A, G, α) on X is called involutive if the representation space X is a Hilbert space, π is an involutive representation of A and U is a unitary representation of G.

We can use Lemma 2.2.9 to obtain a similar general result for normed dynamical systems which, for $G = \{e\}$, specializes to Lemma 2.2.9 again.

Lemma 2.2.13. *Let (A, G, α) be a normed algebra dynamical system, where A has a bounded approximate left identity. Let (π, U) be a covariant representation of (A, G, α) on the Banach space X, and assume that π is bounded.*

(i) If (π, U) is non-degenerate and Z is a subspace which is invariant under both $\pi(A)$ and $U(G)$, then the restricted covariant representation of (A, G, α) to Z is non-degenerate.

(ii) There is a largest subspace which is invariant under both $\pi(A)$ and $U(G)$ such that the restricted covariant representation of (A, G, α) to it is non-degenerate. This subspace is closed. In fact, it is $\overline{\pi(A) \cdot X}$.

Proof. The first part follows directly from the first part of Lemma 2.2.9. As for the second part, the second part of Lemma 2.2.9 shows that any subspace which is invariant under both $\pi(A)$ and $U(G)$ is contained in $\overline{\pi(A) \cdot X}$. It also yields that π restricted to this space is a non-degenerate representation of A, hence we need only show that it is invariant under $U(G)$. As to this, if $y = \pi(a)x$, where $a \in A$ and $x \in X$, then, for $r \in G$, using the covariance,

$$U_r y = U_r \pi(a)x = \pi(\alpha_r(a))U_r x \in \pi(A) \cdot X.$$

By continuity, this implies that $\overline{\pi(A) \cdot X}$ is invariant under U_r, for all $r \in G$. $\square$

We conclude this subsection with some terminology about intertwining operators. Let A be an algebra, and let G be a group. Suppose that X and Y are two Banach spaces, and that $\pi : A \to B(X)$ and $\rho : A \to B(Y)$ are two representations of A. Then a bounded operator $\Phi : X \to Y$ is said to be a bounded intertwining operator between π and ρ if $\rho(a) \circ \Phi = \Phi \circ \pi(a)$, for all $a \in A$. A bounded intertwining operator between two group representations is defined similarly. If (A, G, α) is a normed dynamical system, and (π, U) and (ρ, V) are two covariant representations on Banach spaces X and Y, respectively, then a bounded operator $\Phi : X \to Y$ is called an intertwining operator for these covariant representations, if Φ is an intertwining operator for π and ρ, as well as for U and V.

2.2.4 $C_c(G, X)$

We will frequently work with functions in $C_c(G, X)$, where X is a Banach space. The next lemma, for the proof of which we refer to [51, Lemma 1.88], shows that these functions are uniformly continuous.

Lemma 2.2.14. *Let G be a locally compact Hausdorff group and let X be a Banach space. If $f \in C_c(G, X)$ and $\varepsilon > 0$, then there exists a neighbourhood V of $e \in G$ such that either one of $sr^{-1} \in V$ or $s^{-1}r \in V$ implies*

$$\|f(s) - f(r)\| < \varepsilon.$$

Remark 2.2.15. In this paper we will sometimes refer to the so-called inductive limit topology on $C_c(G, X)$. In these cases, we will be concerned with nets $(f_i) \in C_c(G, X)$ that converge to $f \in C_c(G, X)$, in the sense that (f_i) is eventually supported in some fixed compact set $K \subset G$, and that (f_i) converges uniformly to f on G. As explained in [51, Remark 1.86] and [36, Appendix D.2], such a net *is* convergent in the inductive limit topology, but the converse need not be true. However, it is true that a map from $C_c(G, X)$, supplied with the inductive limit topology, to a locally convex space is continuous precisely when it carries nets which converge in the above sense to convergent nets. We will use this fact in the sequel.

The algebraic tensor product $C_c(G) \otimes A$ can be identified with a subspace of $C_c(G, A)$, and the following approximation result will be used on several occasions. We refer to [51, Lemma 1.87] for the proof, from which a part of the formulation in the version below follows.

Lemma 2.2.16. *Let G be a locally compact group, and let X be Banach space. If X_0 is a dense subset of X, then $C_c(G) \otimes X_0$ is a dense subset of $C_c(G, X)$ in the* *inductive limit topology. In fact, it is even true that there exists a sequence* (f_n) *in* $C_c(G) \otimes X_0$*, with all supports contained in a fixed compact subset of* G *and which converges uniformly to* f *on* G*, which implies that* $f_n \to f$ *in the inductive limit topology of* $C_c(G, X)$*.*

2.2.5 Vector valued integration

For vector-valued integration in Banach spaces, we base ourselves on an integral defined by duality. The pertinent definition, as well as the existence, are contained in the next result, for the proof of which we refer to [42, Theorem 3.27] or [51, Lemma 1.91].

Theorem 2.2.17. *Let* G *be a locally compact group, and let* X *be a Banach space with dual space* X'*. Then there is a linear map* $f \mapsto \int_G f(s)ds$ *from* $C_c(G, X)$ *to* X *which is characterized by*

$$\left\langle \int_G f(s)\,ds, x' \right\rangle = \int_G \langle f(s), x' \rangle\,ds, \quad \forall f \in C_c(G, X), \forall x' \in X'. \tag{2.2.1}$$

Remark 2.2.18. If X and Y are Banach spaces, it follows easily that bounded operators from X to Y can be pulled through the integral of the above theorem. If X has a bounded involution this can also be pulled through the integral, since a bounded involution is a bounded conjugate linear map which can be viewed as a bounded operator from X to the conjugate Banach space of X.

For $F \in C_c(G \times G, X)$, it is shown in [51, Proposition 1.102] that, if one integrates out one variable, the resulting function is in $C_c(G, X)$. Applying continuous linear functionals, it then follows easily, analogously to the proof of [51, Proposition 1.105], that for such functions F the vector-valued version of Fubini's theorem is valid.

The integral from Theorem 2.2.17 enables us to integrate compactly supported strongly (and $*$-strongly) continuous operator-valued functions (recall that unitary representations are $*$-strongly continuous by Remark 2.2.6).

Proposition 2.2.19. *Let* X *be a Banach space, let* G *be a locally compact group, and let* $\psi : G \to B(X)$ *be compactly supported and strongly continuous. Define*

$$\int_G \psi(s)\,ds := \left[x \mapsto \int_G \psi(s)x\,ds \right], \tag{2.2.2}$$

where the integral on the right hand side is the integral from Theorem 2.2.17. Then $\int_G \psi(s)\,ds \in B(X)$*, and*

$$\left\| \int_G \psi(s) \right\|\,ds \leq \int_G \|\psi(s)\|\,ds. \tag{2.2.3}$$

If $T, R \in B(X)$*, then*

$$T \int_G \psi(s)\,ds\,R = \int_G T\psi(s)R\,ds. \tag{2.2.4}$$

Furthermore, if X is a Hilbert space and ψ is $*$-strongly continuous, then

$$\left(\int_G \psi(s)\,ds\right)^* = \int_G \psi(s)^*\,ds. \tag{2.2.5}$$

Proof. By applying elements of X and functionals we obtain (2.2.4), while (2.2.3) follows from applying elements of X and taking norms. As for (2.2.5), let $x, y \in X$ be arbitrary, then

$$\begin{aligned}\left\langle x, \left(\int_G \psi(s)\,ds\right)^* y\right\rangle &= \left\langle \left(\int_G \psi(s)\,ds\right) x, y\right\rangle \\ &= \int_G \langle \psi(s)x, y\rangle\,ds \\ &= \int_G \langle x, \psi(s)^* y\rangle\,ds \\ &= \left\langle x, \left(\int_G \psi(s)^*\,ds\right) y\right\rangle,\end{aligned}$$

where the $*$-strong continuity of ψ ensures that the last line is well defined. Since this holds for all $x, y \in X$, (2.2.5) follows. □

2.2.6 Quotients

The following standard type result, with a routine proof, will be used many times over, often implicitly. If (D, σ) and (E, τ) are seminormed spaces, a linear map $T : D \to E$, is said to be bounded if there exists $C \geq 0$ such that $\tau(Tx) \leq C\sigma(x)$, for all $x \in D$. The seminorm (which is a norm if τ is a norm) of T, is then defined to be the minimal such C.

Lemma 2.2.20. *Let (D, σ) and (E, τ) be seminormed spaces, let $\overline{D/\ker(\sigma)}^{\sigma}$ be the completion of $D/\ker(\sigma)$ in the norm induced by σ, and let $\overline{E/\ker(\tau)}^{\tau}$ be defined similarly. Suppose that $T : D \to E$ is a bounded linear map. Then $T[\ker(\sigma)] \subset \ker(\tau)$ and, with q^σ and q^τ denoting the canonical maps, there exists a unique bounded operator $\widetilde{T} : \overline{D/\ker(\sigma)}^{\sigma} \to \overline{E/\ker(\tau)}^{\tau}$, such that the diagram*

$$\begin{array}{ccc} D & \xrightarrow{T} & E \\ {\scriptstyle q^\sigma}\downarrow & & \downarrow{\scriptstyle q^\tau} \\ \overline{D/\ker(\sigma)}^{\sigma} & \xrightarrow[\widetilde{T}]{} & \overline{E/\ker(\tau)}^{\tau} \end{array} \tag{2.2.6}$$

is commutative. The norm of $\widetilde{T}$ then equals the seminorm of T. In particular, if (E, τ) is a Banach space, then $T \mapsto \widetilde{T}$ is a Banach space isometry between the bounded operators from D into E and the bounded operators from $\overline{D/\ker(\sigma)}^{\sigma}$ into E.

If, in addition, (D, σ) *is a seminormed algebra,* E *is a Banach algebra, and* T *is a bounded algebra homomorphism, then* $\ker(\sigma)$ *is a two-sided ideal,* $\overline{D/\ker(\sigma)}^{\sigma}$ *is a Banach algebra, and* $\widetilde{T}$ *is a bounded algebra homomorphism. In particular, if* X *is a Banach space and* $T : D \to B(X)$ *is a bounded representation, then* $\widetilde{T}$ *is a bounded representation of* $\overline{D/\ker(\sigma)}^{\sigma}$*. In this case,* T *is non-degenerate if and only if* $\widetilde{T}$ *is non-degenerate, a closed subspace of* X *is invariant for* T *if and only if it is invariant for* $\widetilde{T}$*, and if* Y *is a Banach space,* $\Phi \in B(X, Y)$ *and* $S : D \to B(Y)$ *is a bounded representation, then* Φ *intertwines* T *and* S *if and only if* Φ *intertwines* $\widetilde{T}$ *and* $\widetilde{S}$*.*

Alternatively, if, in addition, D *is an algebra with an involution,* σ *is a* C^**-seminorm,* E *is a Banach algebra with a (possibly unbounded) involution, and* T *is a bounded involutive algebra homomorphism, then* $\ker(\sigma)$ *is a self-adjoint two-sided ideal,* $\overline{D/\ker(\sigma)}^{\sigma}$ *is a* C^**-algebra, and* $\widetilde{T}$ *is a bounded involutive algebra homomorphism.*

2.3 Crossed product: construction and basic properties

Let (A, G, α) be a Banach algebra dynamical system, and let $\mathcal{R}$ be a class of (possibly degenerate) continuous covariant representations of (A, G, α) on Banach spaces. We will construct a Banach algebra, denoted $(A \rtimes_\alpha G)^{\mathcal{R}}$, which deserves to be called the crossed product associated with (A, G, α) and $\mathcal{R}$, and establish some basic properties. As will become clear from the discussion below, an additional condition on $\mathcal{R}$ is needed (see Definition 2.3.1), which is automatic, hence "not visible", in the case of crossed product C^*-algebras. In later sections, we will also require the elements of $\mathcal{R}$ to be non-degenerate, but for the moment this is not necessary.

We now start with the construction. Analogously to the C^*-algebra case, this construction is based on the vector space $C_c(G, A)$, as follows.

Let $f, g \in C_c(G, A)$. As a consequence of Lemma 2.2.11, the function

$$(s, r) \mapsto f(r)\alpha_r(g(r^{-1}s))$$

is in $C_c(G \times G, A)$. Therefore, if $s \in G$, Remark 2.2.18 implies that

$$[f * g](s) := \int_G f(r)\alpha_r(g(r^{-1}s))\, dr$$

is a well-defined element of A, and that the thus defined map $f * g : G \to A$, the twisted convolution product of f and g, is in $C_c(G, A)$. The associativity of this product on $C_c(G, A)$ is easily shown using Fubini, and thus $C_c(G, A)$ has the structure of an associative algebra. An easy computation shows that $\operatorname{supp}(f * g)$ is contained in $\operatorname{supp}(f) \cdot \operatorname{supp}(g)$.

Furthermore, if (A, G, α) is an involutive Banach algebra dynamical system, then the formula

$$f^*(s) := \Delta(s^{-1})\alpha_s(f(s^{-1})^*) \tag{2.3.1}$$

defines an involution on $C_c(G, A)$, so that $C_c(G, A)$ becomes an involutive algebra. The proof of this fact relies on a computation as in [51, page 48], which, as the reader may verify, is valid again because the involution on A is bounded and hence can be pulled through the integral by Remark 2.2.18.

Our next step is to find an algebra seminorm on $C_c(G, A)$. It is here that a substantial difference with the construction as in [51] for crossed products associated with C^*-algebras occurs, leading to Definition 2.3.1.

To start with, assume that (π, U) is a continuous covariant representation in a Banach space X. For $f \in C_c(G, A)$, the function $s \mapsto \pi(f(s))U_s$ is strongly continuous from G into $B(X)$ by continuity of multiplication in the strong operator topology on uniformly bounded subsets. Therefore we can define

$$\pi \rtimes U(f) := \int_G \pi(f(s))U_s \, ds, \tag{2.3.2}$$

where the integral on the right-hand side is as in (2.2.2). Note that if (π, U) is involutive, U is $*$-strongly continuous by Remark 2.2.6, so $s \mapsto \pi(f(s))U_s$ is $*$-strongly continuous by continuity of multiplication in the $*$-strong operator topology on uniformly bounded subsets, hence the involution can be pulled through the integral by (2.2.5). Therefore the computations in the proof of [51, Proposition 2.23] are valid, and they show that $\pi \rtimes U$, called the integrated form of (π, U), is a representation of $C_c(G, A)$, and that it is involutive if (A, G, α) is involutive and (π, U) is an involutive continuous covariant representation.

With the construction as in [51] as a model, the natural way to construct a normed algebra from the associative algebra $C_c(G, A)$, given a collection $\mathcal{R}$ of (possibly degenerate) continuous covariant representations on Banach spaces of the Banach algebra dynamical system (A, G, α), is then as follows. For a continuous covariant representation (π, U) and $f \in C_c(G, A)$, we define $\sigma_{(\pi,U)}(f) := \|\pi \rtimes U(f)\|$. Since $\pi \rtimes U$ is a representation of $C_c(G, A)$, the map $\sigma_{(\pi,U)} : C_c(G, A) \to [0, \infty)$ is an algebra seminorm on $C_c(G, A)$. Moreover, if (π, U) is an involutive continuous covariant representation, then $\pi \rtimes U$ is involutive and hence $\sigma_{(\pi,U)}$ is a C^*-seminorm on $C_c(G, A)$.

Lemma 2.2.2 shows that U is bounded on compact sets K by constants $M_K(U)$, and so we obtain the estimate

$$\sigma_{(\pi,U)}(f) = \left\| \int_G \pi(f(s))U_s \, ds \right\| \leq \|\pi\| \, M_{\mathrm{supp}\,(f)}(U) \, \|f\|_{L^1(G,A)} . \tag{2.3.3}$$

Now in the case of C^*-algebra dynamical systems, one takes the supremum over all such C^*-seminorms corresponding to (non-degenerate) involutive continuous covariant representations (π, U), and one defines the corresponding crossed product as the completion of $C_c(G, A)$ with respect to this seminorm (which can then be shown to be a norm). This is meaningful: since the constants $\|\pi\|$ and $M_{\mathrm{supp}\,(f)}(U)$ in (2.3.3) are then always equal to 1, regardless of the choice of (π, U), the supremum is, indeed, pointwise finite. In general situations, when one wants to construct, in a similar way, a crossed product associated with a given class $\mathcal{R}$ of continuous covariant representations, this supremum over the class need no longer be pointwise finite. Furthermore, even if the supremum *is* well-defined, this supremum seminorm need not be a norm on $C_c(G, A)$. The solution to these problems is, obviously, to consider only classes $\mathcal{R}$ such that there is a uniform bound for π and $M_{\mathrm{supp}\,(f)}(U)$ in (2.3.3), as (π, U) ranges over $\mathcal{R}$, and to use a quotient in the construction, as in Lemma 2.2.20. This leads to the following definitions.

Definition 2.3.1. Let (A, G, α) be a Banach algebra dynamical system, and suppose $\mathcal{R}$ is a class of continuous covariant representations of (A, G, α). Then $\mathcal{R}$ is called *uniformly bounded* if there exist a constant $C \geq 0$ and a function $\nu : G \to [0, \infty)$, which is bounded on compact subsets of G, such that, for all (π, U) in $\mathcal{R}$, $\|\pi\| \leq C$ and $\|U_r\| \leq \nu(r)$, for all $r \in G$.

If $\mathcal{R}$ is a non-empty uniformly bounded class of continuous covariant representations, we let $C^{\mathcal{R}} = \sup_{(\pi,U)\in\mathcal{R}} \|\pi\|$ be the minimal such C, and we let

$$\nu^{\mathcal{R}}(r) := \sup_{(\pi,U)\in\mathcal{R}} \|U_r\|\,, \tag{2.3.4}$$

where $r \in G$, be the minimal such ν.[6]

If (A, G, α) is involutive, then $\mathcal{R}$ is said to be involutive if (π, U) is involutive for all $(\pi, U) \in \mathcal{R}$.

Definition 2.3.2. Let (A, G, α) be a Banach algebra dynamical system, and suppose $\mathcal{R}$ is a non-empty uniformly bounded class of continuous covariant representations. Then we define the algebra seminorm $\sigma^{\mathcal{R}}$ on $C_c(G, A)$ by

$$\sigma^{\mathcal{R}}(f) := \sup_{(\pi,U)\in\mathcal{R}} \|\pi \rtimes U(f)\|\,,$$

for $f \in C_c(G, A)$, and we let the corresponding crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ be the completion of $C_c(G, A)/\ker(\sigma^{\mathcal{R}})$ in the norm $\|\,.\,\|^{\mathcal{R}}$ induced by $\sigma^{\mathcal{R}}$. Multiplication in $(A \rtimes_\alpha G)^{\mathcal{R}}$ will still be denoted by $*$.

Remark 2.3.3.

(*i*) From now on, all representations are assumed to be on Banach spaces rather than on normed spaces, since this is needed when integrating.

(*ii*) Obviously, $\sigma^{\mathcal{R}}$ in Definition 2.3.2 is indeed finite, since, as in (2.3.3),

$$\sigma^{\mathcal{R}}(f) \leq C^{\mathcal{R}} \left(\sup_{r\in\mathrm{supp}\,(f)} \nu^{\mathcal{R}}(r) \right) \|f\|_1 < \infty,$$

for $f \in C_c(G, A)$.

[6] Since $r \mapsto \|U_r\|$ is the supremum of continuous functions, it is a lower semicontinuous function on G, and hence the same holds for $\nu^{\mathcal{R}}$.

(iii) By construction, $(A \rtimes_\alpha G)^{\mathcal{R}}$ is a Banach algebra. If (A, G, α) and $\mathcal{R}$ are both involutive, then the seminorms $\sigma_{(\pi,U)}$, for $(\pi, U) \in \mathcal{R}$, are all C^*-seminorms on $C_c(G, A)$, and hence the same holds for their supremum $\sigma^{\mathcal{R}}$. As has already been observed in Lemma 2.2.20, this implies that $(A \rtimes_\alpha G)^{\mathcal{R}}$ is then a C^*-algebra.

If (A, G, α) is a Banach algebra dynamical system, and $\mathcal{R}$ is a non-empty uniformly bounded class of continuous covariant representations, then the corresponding quotient map from Lemma 2.2.20 will be denoted by $q^{\mathcal{R}}$, rather than $q^{\sigma^{\mathcal{R}}}$. Hence

$$q^{\mathcal{R}} : C_c(G, A) \to (A \rtimes_\alpha G)^{\mathcal{R}}$$

is the quotient homomorphism. Likewise, if E is Banach space, and the linear maps $T : C_c(G, A) \to E$ and $S : C_c(G, A) \to C_c(G, A)$ are $\sigma^{\mathcal{R}}$-bounded, then their norms will be denoted by $\|T\|^{\mathcal{R}}$ and $\|S\|^{\mathcal{R}}$, and the corresponding bounded operators from Lemma 2.2.20 will be denoted by $T^{\mathcal{R}}$ and $S^{\mathcal{R}}$, with norms $\left\|T^{\mathcal{R}}\right\| = \|T\|^{\mathcal{R}}$ and $\left\|S^{\mathcal{R}}\right\| = \|S\|^{\mathcal{R}}$. Hence $T^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to E$ and $S^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to (A \rtimes_\alpha G)^{\mathcal{R}}$ are determined by

$$T^{\mathcal{R}}(q^{\mathcal{R}}(f)) = T(f), \quad S^{\mathcal{R}}(q^{\mathcal{R}}(f)) = q^{\mathcal{R}}(S(f)), \tag{2.3.5}$$

for all $f \in C_c(G, A)$.

If $(\pi, U) \in \mathcal{R}$ is a continuous covariant representation in the Banach space X, then $\pi \rtimes U : C_c(G, A) \to B(X)$ is certainly $\sigma^{\mathcal{R}}$ bounded, with norm at most 1. Hence there is a corresponding contractive representation $(\pi \rtimes U)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(X)$ of the Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ in X, determined by

$$(\pi \rtimes U)^{\mathcal{R}}(q^{\mathcal{R}}(f)) = \pi \rtimes U(f), \tag{2.3.6}$$

for all $f \in C_c(G, A)$. If (A, G, α) and $\mathcal{R}$ are involutive, and X is the Hilbert representation space for $(\pi, U) \in \mathcal{R}$, then $(\pi \rtimes U)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(X)$ is an involutive representation of the C^*-algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ in the Hilbert space X. It is contractive by construction, although this is of course also automatic.

Suppose that $\mathcal{R}$ is a uniformly bounded class of continuous covariant representations. By construction, we have, for $f \in C_c(G, A)$,

$$\left\|q^{\mathcal{R}}(f)\right\|^{\mathcal{R}} = \sigma^{\mathcal{R}}(f) = \sup_{(\pi,U)\in\mathcal{R}} \|\pi \rtimes U(f)\| = \sup_{(\pi,U)\in\mathcal{R}} \left\|(\pi \rtimes U)^{\mathcal{R}}(q^{\mathcal{R}}(f))\right\|.$$

For later use, we establish that this formula for the norm in $(A \rtimes_\alpha G)^{\mathcal{R}}$ extends from $q^{\mathcal{R}}(C_c(G, A))$ to the whole crossed product. The separation property that is immediate from it, will later find a parallel for the left centralizer algebra $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ of $(A \rtimes_\alpha G)^{\mathcal{R}}$ in Proposition 2.6.2, under the extra conditions that A has a bounded left approximate identity and that $\mathcal{R}$ consists of non-degenerate continuous covariant representations only.

Proposition 2.3.4. *Let (A, G, α) be a Banach algebra dynamical system and $\mathcal{R}$ a non-empty uniformly bounded class of continuous covariant representations. Then, for all $c \in (A \rtimes_\alpha G)^{\mathcal{R}}$,*

$$\|c\|^{\mathcal{R}} = \sup_{(\pi,U)\in\mathcal{R}} \left\|(\pi \rtimes U)^{\mathcal{R}}(c)\right\|.$$

In particular, the representations $(\pi \rtimes U)^{\mathcal{R}}$, for $(\pi, U) \in \mathcal{R}$, separate the points of $(A \rtimes_\alpha G)^{\mathcal{R}}$.

Proof. Let $c \in (A \rtimes_\alpha G)^{\mathcal{R}}$. The contractivity of $(\pi \rtimes U)^{\mathcal{R}}$, for $(\pi, U) \in \mathcal{R}$, yields $\sup_{(\pi,U)\in\mathcal{R}} \left\|(\pi \rtimes U)^{\mathcal{R}}(c)\right\| \leq \|c\|^{\mathcal{R}}$.

As for the other inequality, let $\varepsilon > 0$. Choose an $f \in C_c(G, A)$ such that $\left\|c - q^{\mathcal{R}}(f)\right\|^{\mathcal{R}} < \varepsilon/3$, and pick $(\pi, U) \in \mathcal{R}$ such that $\|\pi \rtimes U(f)\| > \left\|q^{\mathcal{R}}(f)\right\|^{\mathcal{R}} - \varepsilon/3$. Then

$$\begin{aligned}
\left\|(\pi \rtimes U)^{\mathcal{R}}(c)\right\| &\geq \left\|(\pi \rtimes U)^{\mathcal{R}}(q^{\mathcal{R}}(f))\right\| - \left\|(\pi \rtimes U)^{\mathcal{R}}(c - q^{\mathcal{R}}(f))\right\| \\
&\geq \|\pi \rtimes U(f)\| - \left\|c - q^{\mathcal{R}}(f)\right\|^{\mathcal{R}} \\
&\geq \left\|q^{\mathcal{R}}(f)\right\|^{\mathcal{R}} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} \\
&\geq \|c\|^{\mathcal{R}} - \frac{\varepsilon}{3} - \frac{2\varepsilon}{3}.
\end{aligned}$$

Therefore, $\sup_{(\pi,U)\in\mathcal{R}} \left\|(\pi \rtimes U)^{\mathcal{R}}(c)\right\| > \|c\|^{\mathcal{R}} - \varepsilon$ for all $\varepsilon > 0$, as desired. □

We will now proceed to show that $q^{\mathcal{R}}(C_c(G) \otimes A)$ is dense in $(A \rtimes_\alpha G)^{\mathcal{R}}$, which will obviously be convenient in later proofs. We start with a lemma which is of some interest in itself.

Lemma 2.3.5. *Let (A, G, α) be a Banach algebra dynamical system.*

(i) If $\mathcal{R}$ is a non-empty uniformly bounded class of continuous covariant representations, then $q^{\mathcal{R}} : C_c(G, A) \to (A \rtimes_\alpha G)^{\mathcal{R}}$ is continuous in the inductive limit topology of $C_c(G, A)$. That is, if $(f_i) \in C_c(G, A)$ is a net, eventually supported in a compact set and converging uniformly to $f \in C_c(G, A)$ on G, then $\sigma^{\mathcal{R}}(f_i - f) \to 0$.

(ii) If (π, U) is a continuous covariant representation, then $\pi \rtimes U$ is continuous in the inductive limit topology. That is, if $(f_i) \in C_c(G, A)$ is a net, eventually supported in a compact set and converging uniformly to $f \in C_c(G, A)$ on G, then $\|\pi \rtimes U(f_i) - \pi \rtimes U(f)\| \to 0$.

Proof. (i) It suffices to prove the case when $f = 0$. Let K be a compact set and i_0 an index such that f_i is supported in K for all $i \geq i_0$. Let M denote an upper bound for $\nu^{\mathcal{R}}$ on K. Then, for $(\pi, U) \in \mathcal{R}$ and $i \geq i_0$,

$$\begin{aligned}\|\pi \rtimes U(f_i)\| &= \left\| \int_G \pi(f_i(s)) U_s \, ds \right\| \\ &\leq \int_K \|\pi\| \, \|f_i(s)\| \, M \, ds \\ &\leq C^{\mathcal{R}} M \mu(K) \, \|f_i\|_\infty .\end{aligned}$$

It follows that, for $i \geq i_0$,

$$\sigma^{\mathcal{R}}(f_i) = \sup_{(\pi,U)\in\mathcal{R}} \|\pi \rtimes U(f_i)\| \leq C^{\mathcal{R}} M \mu(K) \, \|f_i\|_\infty .$$

Hence $\sigma^{\mathcal{R}}(f_i) \to 0$.

(*ii*) This follows from (*i*) by taking $\mathcal{R} = \{(\pi, U)\}$, since then $\sigma^{\mathcal{R}}(f) = \|\pi \rtimes U(f)\|$. □

Corollary 2.3.6. *Let (A, G, α) be a Banach algebra dynamical system and $\mathcal{R}$ a non-empty uniformly bounded class of continuous covariant representations. Then $q^{\mathcal{R}}(C_c(G) \otimes A)$ is dense in $(A \rtimes_\alpha G)^{\mathcal{R}}$.*

Proof. By [51, Lemma 1.87], $C_c(G) \otimes A$ is dense in $C_c(G, A)$ in the inductive limit topology. The above lemma therefore implies that $q^{\mathcal{R}}(C_c(G) \otimes A)$ is dense in $q^{\mathcal{R}}(C_c(G, A))$. Since the latter is dense in $(A \rtimes_\alpha G)^{\mathcal{R}}$ by construction, the result follows. □

2.4 Approximate identities

In this section we are concerned with the existence of a bounded approximate left identity in $(A \rtimes_\alpha G)^{\mathcal{R}}$. This is a key issue in the formalism, as the existence of a bounded left approximate identity will allow us to apply Theorem 2.6.1 later on to pass from (non-degenerate) representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ to representations of the left centralizer algebra of $(A \rtimes_\alpha G)^{\mathcal{R}}$, from which, in turn, we will be able to obtain a continuous covariant representation of (A, G, α).

If $(A \rtimes_\alpha G)^{\mathcal{R}}$ happens to be a C^*-algebra, as is, e.g., the case in [51], then the existence of a bounded two-sided approximate identity is of course automatic, but for the general case some effort is needed to show that the existence of a bounded left approximate identity in A implies the similar property for $(A \rtimes_\alpha G)^{\mathcal{R}}$. For the present paper, we need only a bounded left approximate identity, but we also consider an approximate right identity for completeness and future use.

We need two preparatory results. The first one, Lemma 2.4.1, is only relevant for the case of a bounded approximate right identity.

Lemma 2.4.1. *Let (A, G, α) be a Banach algebra dynamical system and suppose A has a bounded approximate right identity (u_i). Fix an element $a \in A$ and a compact* *set $K \subset G$. Then for all $\varepsilon > 0$ we can find an index i_0 such that, for all $i \geq i_0$ and $s \in K$,*

$$\|a\alpha_s(u_i) - a\| < \varepsilon.$$

Proof. Let $M \geq 1$ be an upper bound for (u_i). By Lemma 2.2.2 we can choose an upper bound $M_K > 0$ for α on K.

By the continuity of $s \mapsto s^{-1} \mapsto \alpha_{s^{-1}}(a) = \alpha_s^{-1}(a)$, for every $s \in K$ there exists a neighbourhood W_s such that, for all $r \in W_s$,

$$\|\alpha_r^{-1}(a) - \alpha_s^{-1}(a)\| < \frac{\varepsilon}{3M_K M}.$$

Choose a finite subcover $W_{s_1}, \ldots, W_{s_n}$ of K. Then there exists an index i_0 such that $i \geq i_0$ implies

$$\|\alpha_{s_k}^{-1}(a)u_i - \alpha_{s_k}^{-1}(a)\| < \frac{\varepsilon}{3M_K},$$

for every $1 \leq k \leq n$. For $s \in K$, choose k such that $s \in W_{s_k}$. Then, for all $i \geq i_0$,

$$\begin{aligned}
\|a\alpha_s(u_i) - a\| &= \left\|\alpha_s(\alpha_s^{-1}(a)u_i - \alpha_s^{-1}(a))\right\| \\
&\leq M_K \left(\|\alpha_s^{-1}(a)u_i - \alpha_{s_k}^{-1}(a)u_i\| + \|\alpha_{s_k}^{-1}(a)u_i - \alpha_{s_k}^{-1}(a)\|\right. \\
&\quad \left. + \|\alpha_{s_k}^{-1}(a) - \alpha_s^{-1}(a)\|\right) \\
&< M_K \left(\frac{\varepsilon}{3M_K M} \cdot M + \frac{\varepsilon}{3M_K} + \frac{\varepsilon}{3M_K M}\right) \leq \varepsilon.
\end{aligned}$$

□

Actually, the existence of a bounded approximate left (resp. right) identity in $(A \rtimes_\alpha G)^{\mathcal{R}}$ is inferred from the existence of an suitable approximate left (resp. right) identity in $C_c(G, A)$ in the inductive limit topology. This is the subject of the following theorem.

Theorem 2.4.2. *Let (A, G, α) be a Banach algebra dynamical system and let $\mathcal{Z}$ be a neighbourhood basis of e of which all elements are contained in a fixed compact set. For each $V \in \mathcal{Z}$, take a positive $z_V \in C_c(G)$ with support contained in V and integral equal to one. Suppose (u_i) is a bounded approximate left (resp. right) identity of A. Then the net $\big(f_{(V,i)}\big)$, where*

$$f_{(V,i)} := z_V \otimes u_i,$$

directed by $(V, i) \leq (W, j)$ if and only if $W \subset V$ and $i \leq j$, is an approximate left (resp. right) identity of $C_c(G, A)$ in the inductive limit topology. In fact, for all $f \in C_c(G, A)$ the net $\big(f_{(V,i)} * f\big)$ (resp. $\big(f * f_{(V,i)}\big)$) is supported in a fixed compact set and converges uniformly to f on G.

Proof. Let K be a compact set containing all V in $\mathcal{Z}$, and assume that (u_i) is bounded by $M > 0$.

Since the $f_{(V,i)}$ are all supported in K, all $f_{(V,i)} * f$ (resp. $f * f_{(V,i)}$) are supported in $K \operatorname{supp}(f)$ (resp. $\operatorname{supp}(f)\, K$) for each $f \in C_c(G) \otimes A$.

For the approximating property we start with elements of $C_c(G) \otimes A$. For this it is sufficient to consider elementary tensors, so fix $0 \neq f = z \otimes a$ with $z \in C_c(G)$ and $a \in A$.

First we consider the approximate left identity. Suppose that $\varepsilon > 0$ is given. Let $M_K > 0$ be an upper bound for α on K. Then, for $s \in G$,

$$
\begin{aligned}
\left\| [f_{(V,i)} * f](s) - f(s) \right\| &= \left\| \int_G f_{(V,i)}(r)\alpha_r(f(r^{-1}s))\,dr - f(s) \right\| \\
&= \left\| \int_G z_V(r) z(r^{-1}s) u_i \alpha_r(a)\,dr - z(s)a \right\| \\
&= \left\| \int_G z_V(r) z(r^{-1}s) u_i \alpha_r(a) - z_V(r) z(s) a\,dr \right\| \\
&\leq \int_G z_V(r) \left\| z(r^{-1}s) u_i \alpha_r(a) - z(s)a \right\| \,dr \\
&\leq \int_{\operatorname{supp}(z_V)} z_V(r) \left\| z(r^{-1}s) u_i \alpha_r(a) - z(s) u_i \alpha_r(a) \right\| \,dr \\
&\quad + \int_{\operatorname{supp}(z_V)} z_V(r) \left\| z(s) u_i \alpha_r(a) - z(s) u_i a \right\| \,dr \\
&\quad + \int_{\operatorname{supp}(z_V)} z_V(r) \left\| z(s) u_i a - z(s) a \right\| \,dr \\
&\leq M M_K \left\| a \right\| \int_{\operatorname{supp}(z_V)} z_V(r) |z(r^{-1}s) - z(s)|\,dr \\
&\quad + \|z\|_\infty M \int_{\operatorname{supp}(z_V)} z_V(r) \left\| \alpha_r(a) - a \right\| \,dr \\
&\quad + \|z\|_\infty \int_{\operatorname{supp}(z_V)} z_V(r) \left\| u_i a - a \right\| \,dr.
\end{aligned}
$$

By the uniform continuity of z, there exists a neighbourhood U_1 of e such that $|z(r^{-1}s) - z(s)| < \varepsilon/(3MM_K \|a\|)$, for all $r \in U_1$ and $s \in G$. Hence, for all $s \in G$, the first term is less than $\varepsilon/3$ as soon as $V \subset U_1$. By the strong continuity of α, there exists a neighbourhood U_2 of e such that $\|\alpha_r(a) - a\| < \varepsilon/(3M\|z\|_\infty)$, for all $r \in U_2$. Hence, for all $s \in G$, the second term is less than $\varepsilon/3$ as soon as $V \subset U_2$. There exists an index i_0 such that the third term is less than $\varepsilon/3$ for all $i \geq i_0$ and all $V \in \mathcal{Z}$. Choose $V_0 \in \mathcal{Z}$ such that $V_0 \subset U_1 \cap U_2$. Then, if $(V,i) \geq (V_0, i_0)$, we have $\left\| f_{(V,i)} * f(s) - f(s) \right\| < \varepsilon$ for all $s \in G$ as required.

The approximate right identity is somewhat more involved. Suppose that $\varepsilon > 0$ is given. Let K_1 be a compact set containing all V in $\mathcal{Z}$, as well as the supports of all $f * f_{(V,i)}$ and f. Let $M_{K_1 K^{-1}} > 0$ be an upper bound for α on $K_1 K^{-1}$. Since

$f(s) * f_{(V,i)}(s) - f(s) = 0$ for $s \notin K_1$, we assume that $s \in K_1$, and then

$$
\begin{aligned}
&\big\|[f * f_{(V,i)}](s) - f(s)\big\| \\
&= \left\| \int_G f(r)\alpha_r(f_{(V,i)}(r^{-1}s))\,dr - f(s) \right\| \\
&= \left\| \int_G z(r)az_V(r^{-1}s)\alpha_r(u_i)\,dr - z(s)a \right\| \\
&= \left\| \int_G z(sr)z_V(r^{-1})a\alpha_{sr}(u_i)\,dr - z(s)a \right\| \\
&= \left\| \int_G \Delta(r^{-1})z(sr^{-1})z_V(r)a\alpha_{sr^{-1}}(u_i) - z_V(r)z(s)a\,dr \right\| \\
&\leq \int_G z_V(r)\left\|\Delta(r^{-1})z(sr^{-1})a\alpha_{sr^{-1}}(u_i) - z(s)a\right\|\,dr \\
&\leq \int_{\mathrm{supp}\,(z_V)} z_V(r)\left\|\Delta(r^{-1})z(sr^{-1})a\alpha_{sr^{-1}}(u_i) - z(sr^{-1})a\alpha_{sr^{-1}}(u_i)\right\|\,dr \\
&\quad + \int_{\mathrm{supp}\,(z_V)} z_V(r)\left\|z(sr^{-1})a\alpha_{sr^{-1}}(u_i) - z(s)a\alpha_{sr^{-1}}(u_i)\right\|\,dr \\
&\quad + \int_{\mathrm{supp}\,(z_V)} z_V(r)\left\|z(s)a\alpha_{sr^{-1}}(u_i) - z(s)a\right\|\,dr \\
&\leq \|z\|_\infty\,\|a\|\,M_{K_1K^{-1}}M \int_{\mathrm{supp}\,(z_V)} z_V(r)|\Delta(r^{-1}) - 1|\,dr \\
&\quad + \|a\|\,M_{K_1K^{-1}}M \int_{\mathrm{supp}\,(z_V)} z_V(r)|z(sr^{-1}) - z(s)|\,dr \\
&\quad + \|z\|_\infty \int_{\mathrm{supp}\,(z_V)} z_V(r)\,\|a\alpha_{sr^{-1}}(u_i) - a\|\,dr.
\end{aligned}
$$

By the continuity of Δ there exists a neighbourhood U_1 of e such that

$$|\Delta(r^{-1}) - 1| < \varepsilon/(3\|z\|_\infty\,\|a\|\,M_{K_1K^{-1}}M),$$

for all $r \in U_1$. Hence, for all $s \in K_1$, the first term is less than $\varepsilon/3$ as soon as $V \subset U_1$. By the uniform continuity of z, there exists neighbourhood U_2 of e such that $|z(sr^{-1}) - z(s)| < \varepsilon/(3\,\|a\|\,M_{KK_1^{-1}}M)$, for all $r \in U_2$ and $s \in G$. Hence, for all $s \in K_1$, the second term is less than $\varepsilon/3$ as soon as $V \subset U_2$. An application of Lemma 2.4.1 to the compact set K_1K^{-1} shows that there exists an index i_0 such that $\|a\alpha_{sr^{-1}}(u_i) - a\| < \varepsilon/(3\|z\|_\infty)$, for all $i \geq i_0$, $s \in K_1$ and $r \in K$. Hence, for all $s \in K_1$, the third term is less than $\varepsilon/3$ if $i \geq i_0$. Choose $V_0 \in \mathcal{Z}$ such that $V_0 \subset U_1 \cap U_2$. Then, if $(V,i) \geq (V_0, i_0)$, we have $\big\|[f * f_{(V,i)}](s) - f(s)\big\| < \varepsilon$ for all $s \in K_1$, and hence for all $s \in G$, as required.

We now pass from $C_c(G) \otimes A$ to $C_c(G,A)$, using that $C_c(G) \otimes A$ is uniformly dense in $C_c(G,A)$ (a rather weak consequence of [51, Lemma 1.87]).

We start with the left approximate identity. Let $f \in C_c(G, A)$ and $\varepsilon > 0$ be given. For arbitrary $g \in C_c(G, A)$ and $s \in G$ we have

$$\begin{aligned}
&\big\|[f_{(V,i)} * f](s) - f(s)\big\| \\
&\leq \big\|[f_{(V,i)} * (f - g)](s)\big\| + \big\|[f_{(V,i)} * g](s) - g(s)\big\| + \|g(s) - f(s)\| \\
&= \left\|\int_G z_V(r) u_i \alpha_r((f - g)(r^{-1}s))\, dr\right\| + \big\|[f_{(V,i)} * g](s) - g(s)\big\| + \|g(s) - f(s)\| \\
&\leq M M_K \|f - g\|_\infty \int_{\operatorname{supp}(z_V)} z_V(r)\, dr + \big\|[f_{(V,i)} * g](s) - g(s)\big\| + \|g - f\|_\infty \\
&\leq (M M_K + 1)\|f - g\|_\infty + \big\|f_{(V,i)} * g - g\big\|_\infty .
\end{aligned}$$

The cited density yields $g \in C_c(G) \otimes A$ such that the first term is less than $\varepsilon/2$. By the first part of the proof, there exists an index (V_0, i_0) such that the second term is less than $\varepsilon/2$ for all $(V, i) \geq (V_0, i_0)$. Therefore $\|f_{(V,i)} * f - f\|_\infty < \varepsilon$ for all $(V, i) \geq (V_0, i_0)$.

As for the approximate right identity, let $f \in C_c(G, A)$ and $\varepsilon > 0$ be given. As above, we let K_1 be a compact set containing all V in $\mathcal{Z}$, as well as the supports of all $f * f_{(V,i)}$ and f, and choose an upper bound $M_{K_1 K^{-1}} > 0$ for α on $K_1 K^{-1}$. Let $N_{K^{-1}}$ be an upper bound for Δ on K^{-1}. Then, for arbitrary $g \in C_c(G, A)$ and $s \in K_1$, we have

$$\begin{aligned}
&\big\|f * f_{(V,i)}(s) - f(s)\big\| \\
&\leq \big\|[(f - g) * f_{(V,i)}](s)\big\| + \big\|[g * f_{(V,i)}](s) - g(s)\big\| + \|g(s) - f(s)\| \\
&= \left\|\int_G (f - g)(r) z_V(r^{-1}s) \alpha_r(u_i)\, dr\right\| + \big\|[g * f_{(V,i)}](s) - g(s)\big\| + \|g(s) - f(s)\| \\
&= \left\|\int_{\operatorname{supp}(z_V)} \Delta(r^{-1})(f - g)(sr^{-1}) z_V(r) \alpha_{sr^{-1}}(u_i)\, dr\right\| \\
&\quad + \big\|[g * f_{(V,i)}](s) - g(s)\big\| + \|g(s) - f(s)\| \\
&\leq N_{K^{-1}} \|f - g\|_\infty M_{K_1 K^{-1}} M \int_{\operatorname{supp}(z_V)} z_V(r)\, dr \\
&\quad + \big\|[g * f_{(V,i)}](s) - g(s)\big\| + \|g - f\|_\infty \\
&\leq (N_{K^{-1}} M_{K_1 K^{-1}} M + 1)\|f - g\|_\infty + \|g * f_{(V,i)} - g\|_\infty.
\end{aligned}$$

As above, there exists an index (V_0, i_0) such that, for all $(V, i) \geq (V_0, i_0)$,

$$\|f * f_{(V,i)}(s) - f(s)\| < \varepsilon$$

for all $s \in K_1$. Since this is trivially true for $s \notin K_1$, we are done. $\square$

After these preparations we can now establish that $(A \rtimes_\alpha G)^{\mathcal{R}}$ has a bounded approximate left identity if A has one. We keep track of the constants rather precisely, since the upper bound for the norms of a bounded approximate left identity of $(A \rtimes_\alpha G)^{\mathcal{R}}$ will enter the picture naturally when considering the relation between (non-degenerate) continuous covariant representations of (A, G, α) and (non-degenerate) bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ later on, cf. Remark 2.8.4 and Section 2.9. Therefore, before we prove the result on the approximate identities, let us introduce the relevant constant.

Definition 2.4.3. Let (A, G, α) be a Banach algebra dynamical system and $\mathcal{R}$ a non-empty uniformly bounded class of continuous covariant representations. Let $\nu^{\mathcal{R}} : G \to [0, \infty)$ be defined as in (2.3.4). Let $\mathcal{Z}$ be a neighbourhood basis of e of which all elements are contained in a fixed compact set, and define

$$N^{\mathcal{R}} = \inf_{V \in \mathcal{Z}} \sup_{r \in V} \nu^{\mathcal{R}}(r) < \infty. \tag{2.4.1}$$

Note that, since all V in $\mathcal{Z}$ are contained in a fixed compact set, and $\nu^{\mathcal{R}}$ is bounded on compacta, $N^{\mathcal{R}}$ is indeed finite. Furthermore, this definition of $N^{\mathcal{R}}$ does not depend on the choice of $\mathcal{Z}$. To see this, let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be two neighbourhood bases as in the theorem. For every $V_1 \in \mathcal{Z}_1$, there exists $V_2 \in \mathcal{Z}_2$ such that $V_2 \subset V_1$, and then $\sup_{r \in V_2} \nu^{\mathcal{R}}(r) \leq \sup_{r \in V_1} \nu^{\mathcal{R}}(r)$. This implies that $\inf_{V \in \mathcal{Z}_2} \sup_{r \in V} \nu^{\mathcal{R}}(r) \leq \inf_{V \in \mathcal{Z}_1} \sup_{r \in V} \nu^{\mathcal{R}}(r)$, and the independence of the choice obviously follows.

The constant $N^{\mathcal{R}}$ can be viewed as $\limsup_{r \to e} \nu^{\mathcal{R}}(r)$.

Theorem 2.4.4. *Let (A, G, α) be a Banach algebra dynamical system, where A has an M-bounded approximate left (resp. right) identity (u_i), and let $\mathcal{R}$ be a non-empty uniformly bounded class of continuous covariant representations. Let $\varepsilon > 0$, and choose a neighbourhood V_0 of e with compact closure such that*

$$N^{\mathcal{R}} \leq \sup_{r \in V_0} \nu^{\mathcal{R}}(r) \leq N^{\mathcal{R}} + \varepsilon.$$

Let $\mathcal{Z}$ be a neighbourhood basis of e of which all elements are contained in V_0. For each $V \in \mathcal{Z}$, let $z_V \in C_c(G)$ be a positive function with support contained in V and integral equal to one. Define $f_{(V,i)} := z_V \otimes u_i$, for each $V \in \mathcal{Z}$ and each index i.

Then the associated net $\left(q^{\mathcal{R}}(f_{(V,i)})\right)$ as above is a $C^{\mathcal{R}} M (N^{\mathcal{R}} + \varepsilon)$-bounded approximate left (resp. right) identity of $(A \rtimes_\alpha G)^{\mathcal{R}}$.

If V_0 satisfies $N^{\mathcal{R}} = \sup_{r \in V_0} \nu^{\mathcal{R}}(r)$, then $\left(q^{\mathcal{R}}(f_{(V,i)})\right)$ is $C^{\mathcal{R}} M N^{\mathcal{R}}$-bounded.

Proof. We will prove the left version, the right version is similar.

If $V \in \mathcal{Z}$ and i is an index then we find that, for $(\pi, U) \in \mathcal{R}$,

$$\begin{aligned} \left\| \pi \rtimes U(f_{(V,i)}) \right\| &= \left\| \int_{V_0} z_V(s) \pi(u_i) U_s \, ds \right\| \\ &\leq \|\pi\| \, \|u_i\| \sup_{r \in V} \|U_r\| \int_V z_V(s) \, ds \\ &\leq C^{\mathcal{R}} M \sup_{r \in V_0} \nu^{\mathcal{R}}(r). \end{aligned} \tag{2.4.2}$$

Hence $\sigma^{\mathcal{R}}(q^{\mathcal{R}}(f_{(V,i)})) = \sigma^{\mathcal{R}}(f_{(V,i)}) \leq C^{\mathcal{R}}M\sup_{r\in V_0}\nu^{\mathcal{R}}(r) \leq C^{\mathcal{R}}M(N^{\mathcal{R}}+\varepsilon)$, as desired. To show that $\big(q^{\mathcal{R}}(f_{(V,i)})\big)$ is actually an approximate left identity, we start by noting that, according to Theorem 2.4.2, $f_{(V,i)} * f \to f$ in the inductive limit topology on $C_c(G,A)$, for all $f\in C_c(G,A)$. Therefore, Lemma 2.3.5 implies that $q^{\mathcal{R}}(f_{(V,i)}) * q^{\mathcal{R}}(f) \to q^{\mathcal{R}}(f)$, for all $f\in C_c(G,A)$. Since we have already established that $\big(q^{\mathcal{R}}(f_{(V,i)})\big)$ is uniformly bounded in $(A\rtimes_\alpha G)^{\mathcal{R}}$, an easy 3ε-argument shows that the net is indeed a left approximate identity of $(A\rtimes_\alpha G)^{\mathcal{R}}$.

As for the second part, if V_0 and $\mathcal{Z}$ are as indicated and $V\subset V_0$ is in $\mathcal{Z}$, then a computation as in (2.4.2) yields

$$\left\|\pi\rtimes U(f_{(V,i)})\right\| \leq C^{\mathcal{R}}M\sup_{r\in V_0}\nu^{\mathcal{R}}(r) \leq C^{\mathcal{R}}M(N^{\mathcal{R}}+\varepsilon),$$

hence $\sigma^{\mathcal{R}}(q^{\mathcal{R}}(f_{(V,i)})) \leq C^{\mathcal{R}}M(N^{\mathcal{R}}+\varepsilon)$. This computation with $\varepsilon=0$ shows the final remark of the theorem. $\square$

For convenience we introduce the following notation.

Definition 2.4.5. Let $\mathcal{A}$ be a normed algebra with a bounded approximate left (resp. right) identity. Then $M_l^{\mathcal{A}}$ (resp. $M_r^{\mathcal{A}}$) denotes the infimum of the upper bounds of all approximate left (resp. right) identities. If (A,G,α) is a Banach algebra dynamical system, with A having a bounded left (resp. right) approximate identity, and $\mathcal{R}$ is a non-empty uniformly bounded class of continuous covariant representations, then we will write $M_l^{\mathcal{R}}$ (resp. $M_r^{\mathcal{R}}$), rather than $M_l^{(A\rtimes_\alpha G)^{\mathcal{R}}}$ $\Big($resp. $M_r^{(A\rtimes_\alpha G)^{\mathcal{R}}}\Big)$.

Corollary 2.4.6. *Let (A,G,α) be a Banach algebra dynamical system, where A has a bounded approximate left (resp. right) identity (u_i), and let $\mathcal{R}$ be a non-empty uniformly bounded class of continuous covariant representations. Then $(A\rtimes_\alpha G)^{\mathcal{R}}$ has a bounded approximate left (resp. right) identity, and*

$$M_l^{\mathcal{R}} \leq C^{\mathcal{R}}M_l^A N^{\mathcal{R}},$$
$$M_r^{\mathcal{R}} \leq C^{\mathcal{R}}M_r^A N^{\mathcal{R}}.$$

Proof. We prove the left version, the right version being similar. If $\varepsilon>0$, then A has an $M_l^A+\varepsilon$ approximate left identity, so by the above theorem $(A\rtimes_\alpha G)^{\mathcal{R}}$ has a $C^{\mathcal{R}}(M_l^A+\varepsilon)(N^{\mathcal{R}}+\varepsilon)$-bounded approximate left identity, and the result follows. $\square$

2.5 Representations: from (A,G,α) to $(A\rtimes_\alpha G)^{\mathcal{R}}$

Our principal interest lies in the relation between a non-empty uniformly bounded class $\mathcal{R}$ of continuous covariant representations of a Banach algebra dynamical system (A,G,α), and the bounded representations of the associated crossed product $(A\rtimes_\alpha G)^{\mathcal{R}}$. In this section, we study the easiest part of this relation, which is concerned with passing from suitable continuous covariant representations of (A,G,α) to

$\sigma^{\mathcal{R}}$-bounded representations of $C_c(G, A)$, and subsequently to bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$. The other way round, i.e., passing from bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ to continuous covariant representations of (A, G, α), is more involved, and will be taken up in Section 2.7, after the preparatory Section 2.6. At that point, non-degeneracy of representations will become essential, but for the present section this is not necessary yet.

Above, we wrote "suitable" representations, because there are more continuous covariant representations yielding bounded representations of the crossed product, than just those used to construct that crossed product (which yield contractive ones). The relevant terminology is introduced in the following definition.

Definition 2.5.1. Let (A, G, α) be a Banach algebra dynamical system, and let $\mathcal{R}$ be a non-empty uniformly bounded class of continuous covariant representations. A covariant representation (π, U) of (A, G, α) in a Banach space X is called $\mathcal{R}$-*continuous*, if it is continuous, and the homomorphism

$$\pi \rtimes U : C_c(G, A) \to B(X)$$

is $\sigma^{\mathcal{R}}$-bounded.

Remark 2.5.2. It is clear that a continuous covariant representation (π, U) of (A, G, α) is $\mathcal{R}$-continuous if and only if $\pi \rtimes U$ is continuous as an operator from the space $C_c(G, A)$, equipped with the topology induced by the seminorm $\sigma^{\mathcal{R}}$, to $B(X)$, equipped with the norm topology.

By Lemma 2.2.20, an $\mathcal{R}$-continuous covariant representation yields a bounded representation of the Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$, determined by (2.3.5), which gives (2.3.6) again:

$$(\pi \rtimes U)^{\mathcal{R}}(q^{\mathcal{R}}(f)) = \pi \rtimes U(f), \tag{2.5.1}$$

for $f \in C_c(G, A)$. Then $\|\pi \rtimes U\| = \|\pi \rtimes U\|^{\mathcal{R}}$. If, in addition, (A, G, α), $\mathcal{R}$ and (π, U) are involutive, then $(\pi \rtimes U)^{\mathcal{R}}$ is an involutive representation of the C^*-algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$. Of course, returning to the not necessarily involutive case, the continuous covariant representations in $\mathcal{R}$ are certainly $\mathcal{R}$-continuous, and the corresponding representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ are contractive.

Later, in Proposition 2.7.1, we will be able to show that, if $\mathcal{R}$ is a non-empty uniformly bounded class of continuous covariant representations and if A has a bounded left approximate identity, the assignment $(\pi, U) \mapsto (\pi \rtimes U)^{\mathcal{R}}$ is injective on the non-degenerate $\mathcal{R}$-continuous covariant representations. For the moment we are interested in the preservation of non-degeneracy, the set of closed invariant subspaces and the Banach space of intertwining operators under this map. As a first step, we consider these issues for the assignment $(\pi, U) \mapsto \pi \rtimes U$, for a still arbitrary continuous covariant representation (π, U). In order to do this, we introduce four maps which will be very useful later on as well. Two of these (j_A and j_G below) will only be needed in the involutive case.

Proposition 2.5.3. *Let (A,G,α) be a Banach algebra dynamical system. Define maps $i_A, j_A : A \to \mathrm{End}\,(C_c(G,A))$ and $i_G, j_G : G \to \mathrm{End}\,(C_c(G,A))$ by*

$$\begin{aligned}
&[i_A(a)f](s) := af(s), && [j_A(a)f](s) := f(s)\alpha_s(a), \\
&[i_G(r)f](s) := \alpha_r(f(r^{-1}s)), && [j_G(r)f](s) := \Delta(r^{-1})f(sr^{-1}).
\end{aligned} \tag{2.5.2}$$

for $a \in A$, $r \in G$ and $f \in C_c(G,A)$. Then i_A and i_G and homomorphisms and j_A and j_G are anti-homomorphisms. If (π,U) is a continuous covariant representation of (A,G,α), then, for $a \in A$, $r \in G$, and $f \in C_c(G,A)$,

$$\begin{aligned}
&\pi \rtimes U(i_A(a)f) = \pi(a)\circ \pi\rtimes U(f), && \pi \rtimes U(j_A(a)f) = \pi\rtimes U(f)\circ\pi(a), \\
&\pi \rtimes U(i_G(r)f) = U_r \circ \pi\rtimes U(f), && \pi \rtimes U(j_G(r)f) = \pi\rtimes U(f)\circ U_r.
\end{aligned} \tag{2.5.3}$$

It is actually true that i_A and i_G map into the left centralizers of $C_c(G,A)$ and that j_A and j_G map into the right centralizers of $C_c(G,A)$. The former will be shown during the proof of Proposition 2.6.4 and the proof of the latter is similar, cf. Proposition 2.6.5.

Proof. It is easy to check that the maps are (anti-)homomorphisms. Let $a \in A$ and $f \in C_c(G,A)$ and let (π,U) be a continuous covariant representation, then using the covariance,

$$\begin{aligned}
\pi \rtimes U(j_A(a)f) &= \int_G \pi[(j_A(a)f)(s)]U_s\,ds \\
&= \int_G \pi(f(s))\pi(\alpha_s(a))U_s\,ds \\
&= \int_G \pi(f(s))U_s\pi(a)\,ds \\
&= \pi\rtimes U(f)\circ\pi(a),
\end{aligned}$$

and for $r \in G$ we obtain

$$\begin{aligned}
\pi \rtimes U(i_G(r)f) &= \int_G \pi[(i_G(r)f)(s)]U_s\,ds \\
&= \int_G \pi[\alpha_r(f(r^{-1}s))]U_s\,ds \\
&= \int_G \pi[\alpha_r(f(s))]U_rU_s\,ds \\
&= \int_G U_r\pi(f(s))U_s\,ds \\
&= U_r\circ\pi\rtimes U(f).
\end{aligned}$$

The other computations are similar and will be omitted. □

Before we continue we need a preparatory result, in which the version for j_A will not be applied immediately, but will be useful later on.

Lemma 2.5.4. *Let (A, G, α) be a Banach algebra dynamical system with a bounded approximate left (resp. right) identity (u_i), and let $f \in C_c(G, A))$. Then $i_A(u_i)f$ (resp. $j_A(u_i)f$) converges to f in the inductive limit topology of $C_c(G, A)$.*

Proof. Starting with the left version, we note that Lemma 2.2.16 implies easily that it is sufficient to prove the statement for elementary tensors. So let $f = z \otimes a$ be an elementary tensor in $C_c(G) \otimes A$. Then $[i_A(u_i)f](s) = z(s)u_i a$, which converges uniformly to $z(s)a = f(s)$ on G.

As to the right version, again Lemma 2.2.16, when combined with the observation that the operators $\alpha_s \in B(A)$ are uniformly bounded as s ranges over a compact subset of G, implies that it is sufficient to prove the statement for $f = z \otimes a \in C_c(G) \otimes A$. Let $\varepsilon > 0$. Then, for $s \in G$,

$$\|[j_A(u_i)f](s) - f(s)\| = \|z(s)a\alpha_s(u_i) - z(s)a\| \leq |z(s)| \, \|a\alpha_s(u_i) - a\| \, .$$

For $s \notin \operatorname{supp}(z)$, the right hand side is zero, for all i. Lemma 2.4.1 shows that there exists an index i_0 such that, for all $i \geq i_0$, the right hand side is less than ε, for all $s \in \operatorname{supp}(z)$, and $i \geq i_0$. Hence $\|j_A(u_i)f - f\|_\infty \to 0$. Therefore, $j_A(u_i)f \to f$ in the inductive limit topology of $C_c(G, A)$. □

Proposition 2.5.5. *Let (A, G, α) be a Banach algebra dynamical system and let (π, U) be a continuous covariant representation of (A, G, α) on the Banach space X.*

(i) If (π, U) is non-degenerate, then $\pi \rtimes U$ is a non-degenerate representation of $C_c(G, A)$. If A has a bounded approximate left identity, the converse also holds.

(ii) If Y is a closed subspace of X which is invariant for both π and U, then Y is invariant for $\pi \rtimes U$.

(iii) If Y is a Banach space, (ρ, V) a continuous covariant representation on Y and $\Phi : X \to Y$ a bounded intertwining operator between (π, U) and (ρ, V), then Φ is a bounded intertwining operator between $\pi \rtimes U$ and $\rho \rtimes V$. If (π, U) is non-degenerate, the converse also holds.

(iv) If (A, G, α) and (π, U) are involutive, then so is $\pi \rtimes U$.

Proof. *(i)* Suppose $0 \neq x \in X$ is of the form $x = \pi(a)y$, and let $\varepsilon > 0$. By the strong continuity of U there exists a neighbourhood V of e such that $s \in V$ implies that $\|U_s y - y\| < \varepsilon / \|\pi(a)\|$. Let $z \in C_c(G)$ be nonnegative with compact support contained in V and with integral equal to 1. Define

$f := z \otimes a \in C_c(G, A)$, then

$$\begin{aligned}
\|\pi \rtimes U(f)y - x\| &= \left\| \int_G \pi(f(s))U_s y \, ds - \int_G z(s)x \, ds \right\| \\
&\leq \int_G z(s) \, \|\pi(a)U_s y - \pi(a)y\| \, ds \\
&\leq \int_G z(s) \, \|\pi(a)\| \, \|U_s y - y\| \, ds \\
&\leq \int_{\operatorname{supp}(z)} z(s) \, \|\pi(a)\| \, \frac{\varepsilon}{\|\pi(a)\|} \, ds = \varepsilon.
\end{aligned}$$

This implies that $\overline{\pi \rtimes U(C_c(G, A)) \cdot X} \supset \overline{\pi(A) \cdot X}$. Therefore, if π is non-degenerate, then so is $\pi \rtimes U$.

For the converse, let (u_i) be a bounded approximate left identity of A. By Remark 2.2.8 we have to show that $\pi(u_i)x \to x$, for all $x \in X$. By the boundedness of π and (u_i) and an easy 3ε-argument, it is sufficient to show this for x in a dense subset of X. For this we choose $\pi \rtimes U(C_c(G, A)) \cdot X$, which is dense in X by assumption. Let $f \in C_c(G, A)$ and $y \in X$, then using (2.5.3),

$$\pi(u_i)\pi \rtimes U(f)y = \pi \rtimes U(i_A(u_i)f)y \to \pi \rtimes U(f)y$$

by Lemma 2.5.4 and the continuity of $\pi \rtimes U$ in the inductive limit topology (Lemma 2.3.5). Hence $\pi(u_i)x \to x$ for all $x \in \pi \rtimes U(C_c(G, A)) \cdot X$ by linearity.

(ii) If Y is a closed subspace invariant for both π and U, then it is immediate from the properties of our vector-valued integral that Y is also invariant under $\pi \rtimes U(C_c(G, A))$.

(iii) Let $\Phi : X \to Y$ be a bounded intertwining operator between (π, U) and (ρ, V). Then for $x \in X$ and $f \in C_c(G, A)$ we have

$$\begin{aligned}
\Phi \circ \pi \rtimes U(f) &= \Phi \circ \int_G \pi(f(s))U_s \, ds \\
&= \int_G \Phi \, \pi(f(s))U_s \, ds \\
&= \int_G \rho(f(s)) \, \Phi \, U_s \, ds \\
&= \int_G \rho(f(s))V_s \, \Phi \, ds \\
&= \int_G \rho(f(s))V_s \, ds \circ \Phi \\
&= \rho \rtimes V(f) \circ \Phi.
\end{aligned}$$

Conversely, suppose that $\Phi : X \to Y$ is a bounded intertwining operator for $U \rtimes \pi$ and $V \rtimes \rho$ and that (π, U) is non-degenerate. For elements of X of the form $\pi \rtimes U(f)x$, where $x \in X$ and $f \in C_c(G,A)$, we obtain for $r \in G$, using (2.5.3),

$$\begin{aligned}
[\Phi \circ U_r](\pi \rtimes U(f)x) &= [\Phi \circ U_r \circ (\pi \rtimes U)(f)]x \\
&= [\Phi \circ (\pi \rtimes U)(i_G(r)f)]x \\
&= [(\rho \rtimes V)(i_G(r)f) \circ \Phi]x \\
&= [V_r \circ (\rho \rtimes V)(f) \circ \Phi]x \\
&= [V_r \circ \Phi \circ (\pi \rtimes U)(f)]x \\
&= [V_r \circ \Phi](\pi \rtimes U(f)x).
\end{aligned}$$

By (i), $\pi \rtimes U$ is non-degenerate, and so $\Phi \circ U_r$ and $V_r \circ \Phi$ agree on a dense subset of X and hence are equal. Similarly we obtain that $\Phi \circ \pi(a) = \rho(a) \circ \Phi$ for all $a \in A$.

(iv) This has been shown in Section 2.3, following (2.3.2).

□

Together with Lemma 2.2.20 the above proposition immediately leads to most the following.

Theorem 2.5.6. *Let (A,G,α) be a Banach algebra dynamical system, and let $\mathcal{R}$ be a non-empty uniformly bounded class of continuous covariant representations. Consider the assignment $(\pi,U) \to (\pi \rtimes U)^{\mathcal{R}}$ from the $\mathcal{R}$-continuous covariant representations of (A,G,α) to the bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$.*

(i) If (π,U) is non-degenerate, then $(\pi \rtimes U)^{\mathcal{R}}$ is a non-degenerate representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$. If A has a bounded approximate left identity, the converse also holds.

(ii) If Y is a closed subspace of X which is invariant for both π and U, then Y is invariant for $(\pi \rtimes U)^{\mathcal{R}}$. If (π,U) is non-degenerate and A has a bounded approximate left identity, the converse also holds.

(iii) If Y is a Banach space, (ρ,V) a continuous covariant representation on Y and $\Phi : X \to Y$ a bounded intertwining operator between (π,U) and (ρ,V), then Φ is a bounded intertwining operator between $(\pi \rtimes U)^{\mathcal{R}}$ and $(\rho \rtimes V)^{\mathcal{R}}$. If (π,U) is non-degenerate, the converse also holds.

(iv) If (A,G,α), $\mathcal{R}$, and (π,U) are involutive, then so is $(\pi \rtimes U)^{\mathcal{R}}$.

Furthermore, for a general Banach dynamical system, $\|\pi \rtimes U\|^{\mathcal{R}} = \left\|(\pi \rtimes U)^{\mathcal{R}}\right\|$, for each $\mathcal{R}$-continuous covariant representation (π,U) of (A,G,α).

Proof. All that has to be shown is that if Y is invariant for $(\pi \rtimes U)^{\mathcal{R}}$, (π,U) is non-degenerate and A has a bounded approximate left identity, then Y is invariant for (π,U). Under these assumptions, the first part of the theorem shows that $(\pi \rtimes U)^{\mathcal{R}}$ is non-degenerate. By Theorem 2.4.4, $(A \rtimes_\alpha G)^{\mathcal{R}}$ has a bounded approximate left identity, so by Lemma 2.2.9, $(\pi \rtimes U)^{\mathcal{R}}|_Y$ is non-degenerate. Using the density of $q^{\mathcal{R}}(C_c(G,A))$ in $(A \rtimes_\alpha G)^{\mathcal{R}}$, we obtain that

$$\pi \rtimes U(C_c(G,A)) \cdot Y = (\pi \rtimes U)^{\mathcal{R}} \left(q^{\mathcal{R}}(C_c(G,A))\right) \cdot Y$$

is dense in Y. Now for $a \in A$, $r \in G$, $y \in Y$ and $f \in C_c(G,A)$ by (2.5.3),

$$\begin{aligned} \pi(a) \circ \pi \rtimes U(f)y &= \pi \rtimes U(i_A(a)f)y \in Y \\ U_r \circ \pi \rtimes U(f)y &= \pi \rtimes U(i_G(r)f)y \in Y, \end{aligned}$$

so $\pi(a)$ and U_r map a dense subset of Y into Y, which proves the claim. □

2.6 Centralizer algebras

The passage from non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ to continuous covariant representations of (A, G, α) in Section 2.7 will be obtained using the left centralizer algebra of $(A \rtimes_\alpha G)^{\mathcal{R}}$. This will be done in Proposition 2.7.1 below, and it consists of two steps. The idea is to first construct, by general means, a bounded representation of the left centralizer algebra of $(A \rtimes_\alpha G)^{\mathcal{R}}$ from a given non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$, and next to compose this new representation with covariant homomorphisms (to be constructed below) of A and G into this left centralizer algebra, thus obtaining (at least algebraically) a covariant representations of the group and the algebra.

In the present section, which is a preparation for the next, we start by recalling the basic general theorem which underlies the first step in the above procedure. This will make it obvious why it is so important that $(A \rtimes_\alpha G)^{\mathcal{R}}$ has a bounded left approximate identity if A has one, something which is—as observed before– automatic in the C^*-case, but not in the general setting. Next we construct the homomorphisms needed for the second step. We also include some results for the double centralizer algebra of $(A \rtimes_\alpha G)^{\mathcal{R}}$; these will be needed for the involutive case only.

Commencing with representations of a general normed algebra and its centralizer algebras, we let $\mathcal{A}$ be a normed algebra: the results below will be applied with the Banach algebra $\mathcal{A} = (A \rtimes_\alpha G)^{\mathcal{R}}$. We let $\mathcal{M}_l(\mathcal{A}) \subset B(\mathcal{A})$ denotes the unital normed algebra of left centralizers of $\mathcal{A}$, i.e., the algebra of bounded operators $L : \mathcal{A} \to \mathcal{A}$ commuting with all right multiplications, or equivalently, satisfying $L(a)b = L(ab)$ for all $a, b \in \mathcal{A}$. Every $a \in \mathcal{A}$ determines a left centralizer by left multiplication, and we let $\lambda : \mathcal{A} \to \mathcal{M}_l(\mathcal{A})$ denotes the corresponding homomorphism. Likewise, the algebra $\mathcal{M}_r(\mathcal{A}) \subset B(\mathcal{A})$ denotes the unital normed algebra of right centralizers, i.e., the algebra of operators $R : \mathcal{A} \to \mathcal{A}$ commuting with all left multiplications, or equivalently, satisfying $R(ab) = aR(b)$ for all $a, b \in \mathcal{A}$, and $\rho : \mathcal{A} \to \mathcal{M}_r(\mathcal{A})$ denotes the canonical anti-homomorphism. The unital normed algebra of double centralizers of $\mathcal{A}$ is denoted by $\mathcal{M}(\mathcal{A})$ and consists of pairs (L, R), where L is a left centralizer and R is a right centralizer, such that $aL(b) = R(a)b$ for all $a, b \in \mathcal{A}$. Multiplication in $\mathcal{M}(\mathcal{A})$ is defined by $(L_1, R_1)(L_2, R_2) = (L_1L_2, R_2R_1)$ and the norm by $\|(L,R)\|_{\mathcal{M}(\mathcal{A})} = \max(\|L\|, \|R\|)$. We let $\phi_l : \mathcal{M}(\mathcal{A}) \to \mathcal{M}_l(\mathcal{A})$ denote the contractive unital homomorphism $(L,R) \mapsto L$, and $\delta : \mathcal{A} \to \mathcal{M}(\mathcal{A})$ denote the homomorphism $a \mapsto (\lambda(a), \rho(a))$.

If L is invertible in $\mathcal{M}_l(\mathcal{A})$ and R is invertible in $\mathcal{M}_r(\mathcal{A})$, then (L,R) is invertible in $\mathcal{M}(\mathcal{A})$ with inverse $(L,R)^{-1} = (L^{-1}, R^{-1})$. If $\mathcal{A}$ has a bounded involution, then for $L \in \mathcal{M}_l(\mathcal{A})$ the map $L^* : \mathcal{A} \to \mathcal{A}$ defined by $L^*(a) := (L(a^*))^*$ is a right centralizer, and for $R \in \mathcal{M}_r(\mathcal{A})$ the map R^* defined by $R^*(a) := (R(a^*))^*$ is a left centralizer. Furthermore $(L^*)^* = L$ and $(R^*)^* = R$. As a consequence, the map $(L,R) \mapsto (R^*, L^*)$ is a bounded involution on $\mathcal{M}(\mathcal{A})$.

Obviously, if $\mathcal{A}$ is a Banach algebra, then so are $\mathcal{M}_l(\mathcal{A})$, $\mathcal{M}_r(\mathcal{A})$, and $\mathcal{M}(\mathcal{A})$.

In the following theorem we collect a few results from [9, Remark 2.2, Theorem 4.1 and Theorem 4.5]. The constant $M_l^{\mathcal{A}}$ in it is was defined in Definition 2.4.5 as the infimum of the upper bounds of all approximate left identities. The theorem implies, in particular, that, given a non-degenerate bounded Banach space representation of a normed algebra with a bounded approximate left identity, there exists a unique representation (which is then automatically bounded and non-degenerate) of its left centralizer algebra which is compatible with the canonical homomorphism $\lambda : \mathcal{A} \to \mathcal{M}_l(\mathcal{A})$. This is a crucial step in our approach, and it should be thought of as the analogue of extending a representation of a C^*-algebra to its multiplier algebra.

Theorem 2.6.1. *Let $\mathcal{A}$ be a normed algebra with a bounded approximate left identity, and let X be a Banach space.*

If $T : \mathcal{A} \to \mathcal{B}(X)$ is a non-degenerate bounded representation, then there exists a unique homomorphism $\overline{T} : \mathcal{M}_l(\mathcal{A}) \to \mathcal{B}(X)$ such that the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "T"] \arrow[d, "\delta"] \arrow[dr, "\lambda"] & \mathcal{B}(X) \\ \mathcal{M}(\mathcal{A}) \arrow[r, "\phi_l"'] & \mathcal{M}_l(\mathcal{A}) \arrow[u, "\overline{T}"'] \end{tikzcd} \tag{2.6.1}$$

is commutative. All maps in the diagram are bounded homomorphisms, and $\overline{T}$ is unital. One has $\|\overline{T}\| \le M_l^{\mathcal{A}} \|T\|$, which implies $\|\overline{T} \circ \phi_l\| \le M_l^{\mathcal{A}} \|T\|$. In particular, $\overline{T}$ and $\overline{T} \circ \phi_l$ are non-degenerate bounded representations of $\mathcal{M}_l(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$ on X.

The image $T(\mathcal{A})$ is a left ideal in $\overline{T}(\mathcal{M}_l(\mathcal{A}))$. In fact, if $L \in \mathcal{M}_l(\mathcal{A})$ and $a \in \mathcal{A}$, then

$$\overline{T}(L) \circ T(a) = T(L(a)). \tag{2.6.2}$$

If (u_i) is any bounded approximate left identity of $\mathcal{A}$ and if $L \in \mathcal{M}_l(\mathcal{A})$, then for $x \in X$ we have

$$\overline{T}(L)x = \lim_i T(L(u_i))x. \tag{2.6.3}$$

In particular, the set of closed invariant subspaces of T coincides with the set of closed invariant subspaces of $\overline{T}$, and if $S : \mathcal{A} \to B(X)$ is another non-degenerate *bounded representation, then the set of bounded intertwining operators of T and S coincides with the set of bounded intertwining operators of $\overline{T}$ and $\overline{S}$.*

If in addition $\mathcal{A}$ has a bounded involution, X is a Hilbert space and T is involutive, then $\overline{T} \circ \phi_l$ is involutive.

If, returning to our original context, (A, G, α) is a Banach algebra dynamical system, and $\mathcal{R}$ is a non-empty uniformly bounded class of continuous covariant representations, then each (π, U) in $\mathcal{R}$, being obviously $\mathcal{R}$-continuous, yields a bounded (even contractive) representation $(\pi \rtimes U)^{\mathcal{R}}$ of $(A \rtimes_\alpha G)^{\mathcal{R}}$, and if $(\pi, U) \in \mathcal{R}$ is non-degenerate, then $(\pi \rtimes U)^{\mathcal{R}}$ is non-degenerate as well, by Theorem 2.5.6. If, in addition, A has a bounded approximate left identity, then $(A \rtimes_\alpha G)^{\mathcal{R}}$ has a bounded approximate left identity by Corollary 2.4.6, hence Theorem 2.6.1 provides a bounded representation $\overline{(\pi \rtimes U)^{\mathcal{R}}}$ of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$. These representations $\overline{(\pi \rtimes U)^{\mathcal{R}}}$, for $(\pi, U) \in \mathcal{R}$, are used in the following result, which is a parallel of the separation property in Proposition 2.3.4.

Proposition 2.6.2. *Let (A, G, α) be a Banach algebra dynamical system, where A has a bounded approximate left identity, and let $\mathcal{R}$ be a non-empty uniformly bounded class of non-degenerate continuous covariant representations. Then the non-degenerate bounded representations $\overline{(\pi \rtimes U)^{\mathcal{R}}}$ of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$, for $(\pi, U) \in \mathcal{R}$, separate the points of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$.*

Proof. Let $L \in \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ be such that $\overline{(\pi \rtimes U)^{\mathcal{R}}}(L) = 0$, for all $(\pi, U) \in \mathcal{R}$. Then, for arbitrary $c \in (A \rtimes_\alpha G)^{\mathcal{R}}$, the combination of Proposition 2.3.4 and (2.6.2) shows that

$$\begin{aligned}
\|L(c)\|^{\mathcal{R}} &= \sup_{(\pi,U)\in\mathcal{R}} \left\|(\pi \rtimes U)^{\mathcal{R}}(L(c))\right\| \\
&= \sup_{(\pi,U)\in\mathcal{R}} \left\|\overline{(\pi \rtimes U)^{\mathcal{R}}}(L) \circ (\pi \rtimes U)^{\mathcal{R}}(c)\right\| \\
&= 0.
\end{aligned}$$

Hence $L = 0$. □

We continue our preparation for the representation theory in the next section by investigating a particular continuous covariant representation of (A, G, α) in $(A \rtimes_\alpha G)^{\mathcal{R}}$, needed for the second step in the procedure outlined in the beginning of this section. An important feature, in view of Theorem 2.6.1, of this particular continuous covariant representations is that the corresponding images of A and G are contained in the left centralizer algebra $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ of $(A \rtimes_\alpha G)^{\mathcal{R}}$, so that it can be composed with representations of $M_l((A \rtimes_\alpha G)^{\mathcal{R}})$ resulting from the first step. We will now proceed to construct this continuous covariant representations, which is done using the actions of A and G on $C_c(G, A)$, as defined in (2.5.2).

Lemma 2.6.3. *Let (A, G, α) be a Banach algebra dynamical system, and let $\mathcal{R}$ be a non-empty uniformly bounded class of continuous covariant representations. Let*

$a \in A$ and $r \in G$. Then the maps

$$i_A(a),\ i_G(r) : (C_c(G,A), \sigma^{\mathcal{R}}) \to (C_c(G,A), \sigma^{\mathcal{R}})$$

are bounded. In fact,

$$\|i_A(a)\|^{\mathcal{R}} \leq \sup_{(\pi,U)\in\mathcal{R}} \|\pi(a)\| \leq C^{\mathcal{R}} \|a\|,$$

and

$$\|i_G(r)\|^{\mathcal{R}} \leq \nu^{\mathcal{R}}(r).$$

Proof. Let $a \in A$. Then, for $f \in C_c(G,A)$ and $(\pi,U) \in \mathcal{R}_r$, using (2.5.3) in the first step, we find that

$$\begin{aligned}
\|\pi \rtimes U(i_A(a)f)\| &= \|\pi(a) \circ U \rtimes \pi(f)\| \\
&\leq \left(\sup_{(\pi,U)\in\mathcal{R}} \|\pi(a)\|\right)\left(\sup_{(\pi,U)\in\mathcal{R}} \|\pi \rtimes U(f)\|\right) \\
&= \left(\sup_{(\pi,U)\in\mathcal{R}} \|\pi(a)\|\right)\sigma^{\mathcal{R}}(f).
\end{aligned}$$

Taking the supremum over $(\pi,U) \in \mathcal{R}$ implies the statement concerning $i_A(a)$. The statement concerning $i_G(r)$ follows similarly. □

As a consequence of the above proposition and Lemma 2.2.20, the operators $i_A(a)$ and $i_G(r)$ yield bounded operators from $(A \rtimes_\alpha G)^{\mathcal{R}}$ to itself with the same norm. For typographical reasons, we will denote these elements of $B((A \rtimes_\alpha G)^{\mathcal{R}})$ by $i_A^{\mathcal{R}}(a)$ and $i_G^{\mathcal{R}}(r)$ rather than $i_A(a)^{\mathcal{R}}$ and $i_G(r)^{\mathcal{R}}$. Hence, if $a \in A$ and $r \in G$, then $i_A^{\mathcal{R}}(a),\ i_G^{\mathcal{R}}(r) \in B((A \rtimes_\alpha G)^{\mathcal{R}})$ are determined by

$$i_A^{\mathcal{R}}(a)(q^{\mathcal{R}}(f)) = q^{\mathcal{R}}(i_A(a)f), \quad i_G^{\mathcal{R}}(r)(q^{\mathcal{R}}(f)) = q^{\mathcal{R}}(i_G(r)f) \tag{2.6.4}$$

for all $f \in C_c(G,A)$.

In Proposition 2.5.3 we have noted that the maps $i_A : A \to \operatorname{End}(C_c(G,A))$ and $i_G : G \to \operatorname{End}(C_c(G,A))$ are homomorphisms. As a consequence of (2.6.4) and the density of $q^{\mathcal{R}}(C_c(G,A))$ in $(A \rtimes_\alpha G)^{\mathcal{R}}$, the same is then true for the maps $i_A^{\mathcal{R}} : A \to B((A \rtimes_\alpha G)^{\mathcal{R}})$ and $i_G^{\mathcal{R}} : G \to B((A \rtimes_\alpha G)^{\mathcal{R}})$. Hence we have a pair of representations $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ on $(A \rtimes_\alpha G)^{\mathcal{R}}$.

Proposition 2.6.4. *Let (A,G,α) be a Banach algebra dynamical system, and let $\mathcal{R}$ be a non-empty uniformly bounded class of continuous covariant representations. Then $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$, as defined by (2.6.4), is a continuous covariant representation of (A,G,α) in $(A \rtimes_\alpha G)^{\mathcal{R}}$. The images $i_A^{\mathcal{R}}(A)$ and $i_G^{\mathcal{R}}(G)$ are contained in the left centralizer algebra $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ of $(A \rtimes_\alpha G)^{\mathcal{R}}$, so we have*

$$\begin{aligned}
i_A^{\mathcal{R}} &: A \to \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}}) \subset B((A \rtimes_\alpha G)^{\mathcal{R}}), \\
i_G^{\mathcal{R}} &: G \to \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}}) \subset B((A \rtimes_\alpha G)^{\mathcal{R}}).
\end{aligned}$$

For the operator norm in $B((A \rtimes_\alpha G)^{\mathcal{R}})$ the estimates

$$\left\| i_A^{\mathcal{R}}(a) \right\| \leq \sup_{(\pi,U)\in\mathcal{R}} \|\pi(a)\| \leq C^{\mathcal{R}} \|a\|,$$

where $a \in A$, and

$$\left\| i_G^{\mathcal{R}}(r) \right\| \leq \nu^{\mathcal{R}}(r),$$

where $r \in G$, hold.

If, in addition, A has a bounded approximate left identity, then $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ is non-degenerate.

Although it does not follow from the estimates for the operator norm in Proposition 2.6.4, if A has a bounded approximate left identity and all elements of $\mathcal{R}$ are non-degenerate, then it is actually true that $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ is $\mathcal{R}$-continuous, see Theorem 2.7.2. Proving this will require some extra effort, and we will only be able to do so once more information has been obtained about the relation between $\mathcal{R}$-continuous covariant representations of (A, G, α) and bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$.

Proof. We start by proving the covariance of $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$. For this it is sufficient to show that the pair (i_A, i_G) is covariant, i.e., that $[i_G(r) i_A(a) i_G(r)^{-1} f](s) = [i_A(\alpha_r(a)) f](s)$ for all $f \in C_c(G, A)$, $r, s \in G$, and $a \in A$. Indeed,

$$\begin{aligned} [i_G(r) i_A(a) i_G(r)^{-1} f](s) &= \alpha_r[(i_A(a) i_G(r)^{-1} f)(r^{-1}s)] \\ &= \alpha_r[a i_G(r)^{-1} f(r^{-1}s)] \\ &= \alpha_r[a \alpha_{r^{-1}}(f(rr^{-1}s))] \\ &= \alpha_r(a)(f(s)) \\ &= [i_A(\alpha_r(a)) f](s). \end{aligned}$$

We continue by showing that the bounded operators $i_A^{\mathcal{R}}(a)$ and $i_G^{\mathcal{R}}$ on $(A \rtimes_\alpha G)^{\mathcal{R}}$ are left centralizers of the Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$. To see this, let $a \in A$. Then, for $f, g \in C_c(G, A)$ and $s \in G$,

$$\begin{aligned} [i_A(a)(f * g)](s) &= a \int_G f(r) \alpha_r(g(r^{-1}s))\, dr \\ &= \int_G a f(r) \alpha_r(g(r^{-1}s))\, dr \\ &= [(i_A(a) f) * g](s). \end{aligned}$$

So $i_A(a)$ commutes with right multiplication in $C_c(G, A)$. Hence

$$\begin{aligned} i_A^{\mathcal{R}}(a)(q^{\mathcal{R}}(f) * q^{\mathcal{R}}(g)) = i_A^{\mathcal{R}}(a)(q^{\mathcal{R}}(f * g)) &= q^{\mathcal{R}}(i_A(a)(f * g)) \\ &= q^{\mathcal{R}}([i_A(a) f] * g) = q^{\mathcal{R}}(i_A(a) f) * q^{\mathcal{R}}(g) \\ &= [i_A(a)^{\mathcal{R}} q^{\mathcal{R}}(f)] * q^{\mathcal{R}}(g), \end{aligned}$$

for $f, g \in C_c(G, A)$. From the density of $q^{\mathcal{R}}(C_c(G, A)$ in $(A \rtimes_\alpha G)^{\mathcal{R}}$ and the boundedness of $i_A^{\mathcal{R}}(a)$ it then follows that $i_A^{\mathcal{R}}(a)$ is a left centralizer of $(A \rtimes_\alpha G)^{\mathcal{R}}$. As to the other case, let $r \in G$. Then, for $f, g \in C_c(G, A)$ and $s \in G$,

$$\begin{aligned}
[i_G(r)(f * g)](s) &= \alpha_r([f * g](r^{-1}s)) \\
&= \alpha_r\left(\int_G f(t)\alpha_t(g(t^{-1}r^{-1}s))\,dt\right) \\
&= \int_G \alpha_r(f(t))\alpha_{rt}(g((rt)^{-1}s))\,dt \\
&= \int_G \alpha_r(f(r^{-1}t))\alpha_t(g(t^{-1}s))\,dt \\
&= [(i_G(r)f) * g](s).
\end{aligned}$$

So $i_G(r)$ commutes with right multiplication in $C_c(G, A)$. As for $i_A(a)$, it follows that $i_G^{\mathcal{R}}(r)$ is a left centralizer of $(A \rtimes_\alpha G)^{\mathcal{R}}$.

Next, we will show that $i_G^{\mathcal{R}}$ is strongly continuous. In view of the boundedness of $i_G^{\mathcal{R}}$ on compact neighbourhoods of e, Corollary 2.2.5 implies that we only have to show strong continuity of $i_G^{\mathcal{R}}$ in e on a dense subset of $(A \rtimes_\alpha G)^{\mathcal{R}}$. By Corollary 2.3.6, $q^{\mathcal{R}}(C_c(G) \otimes A)$ is dense in $(A \rtimes_\alpha G)^{\mathcal{R}}$, and so it is sufficient to show that $\sigma^{\mathcal{R}}(i_G(r_i)f - f) \to 0$ for all $f \in C_c(G) \otimes A$, whenever $r_i \to e$ in G. By linearity it is sufficient to consider only elements of the form $z \otimes a$ with $z \in C_c(G)$ and $a \in A$. Therefore, fix $z \otimes a$ and let $r_i \to e$. We may assume that the r_i are all contained in a fixed compact set. It is the obvious that the net $(i_G(r_i)(z \otimes a))$ is likewise supported in a fixed compact set, so by Lemma 2.3.5 it suffices to show that $[i_G(r_i)(z \otimes a)](s) - z(s)a \to 0$, uniformly in s. Since

$$\begin{aligned}
\|[i_G(r_i)(z \otimes a)](s) - z(s)a\| &= \left\|z(r_i^{-1}s)\alpha_{r_i}(a) - z(s)a\right\| \\
&\leq \left\|z(r_i^{-1}s)\alpha_{r_i}(a) - z(r_i^{-1}s)a\right\| + \left\|z(r_i^{-1}s)a - z(s)a\right\| \\
&\leq \|z\|_\infty \|\alpha_{r_i}(a) - a\| + \left\|z(r_i^{-1}s) - z(s)\right\| \|a\|,
\end{aligned}$$

this uniform convergence follows from the strong continuity of α and the uniform continuity of z. Together with the discussion preceding the theorem, this concludes the proof that $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ is a continuous covariant representation of (A, G, α) on $(A \rtimes_\alpha G)^{\mathcal{R}}$.

If, in addition, A has a bounded approximate left identity (u_i), then, for each $f \in C_c(G) \otimes A$, Lemma 2.5.4 shows that $i_A(u_i)f \to f$ in the inductive limit topology. As a consequence, $i_A(A) \cdot C_c(G) \otimes A$ is dense in $C_c(G) \otimes A$ in the inductive limit topology. By Lemma 2.3.5, $i_A^{\mathcal{R}}(A) \cdot q^{\mathcal{R}}(C_c(G) \otimes A) = q^{\mathcal{R}}(i_A(A) \cdot C_c(G) \otimes A)$ is dense in $q^{\mathcal{R}}(C_c(G) \otimes A)$. Since the latter is dense in $(A \rtimes_\alpha G)^{\mathcal{R}}$ by Corollary 2.3.6, $i_A^{\mathcal{R}}$ is thus seen to be non-degenerate. □

The above Proposition 2.6.4 is sufficient for the sequel in the case of general Banach algebra dynamical systems. In the involutive case, the left centralizer algebra alone is no longer sufficient, because of the lack of an involutive structure. In that case, we will use the double centralizer algebra, and in order to establish the result for the double centralizer algebra that will eventually be used, we first need the following right-sided version of part of the above theorem.

Proposition 2.6.5. *Let (A, G, α) be a Banach algebra dynamical system and let $\mathcal{R}$ be a non-empty uniformly bounded class of continuous covariant representations. For $a \in A$ and $r \in G$, let $j_A(a)$ and $j_G(r)$ be as in* (2.6.4). *Then the maps*

$$j_A(a),\ j_G(r) : (C_c(G, A), \sigma^{\mathcal{R}}) \to (C_c(G, A), \sigma^{\mathcal{R}})$$

are bounded. Denote the corresponding bounded operators on $(A \rtimes_\alpha G)^{\mathcal{R}}$ by $j_A^{\mathcal{R}}(a)$ and $j_G^{\mathcal{R}}(r)$, determined by $j_A^{\mathcal{R}}(a)(q^{\mathcal{R}}(f)) = q^{\mathcal{R}}(j_A(a)(f))$, for all $f \in C_c(G, A)$, and by $j_G^{\mathcal{R}}(r)(q^{\mathcal{R}}(f)) = q^{\mathcal{R}}(j_G(r)f)$, for all $f \in C_c(G, A)$.

Then $j_A^{\mathcal{R}} : A \to B((A \rtimes_\alpha G)^{\mathcal{R}})$ is a bounded anti-representation of A in $(A \rtimes_\alpha G)^{\mathcal{R}}$, and $j_G^{\mathcal{R}} : G \to B((A \rtimes_\alpha G)^{\mathcal{R}})$ is a strongly continuous anti-representation of G in $(A \rtimes_\alpha G)^{\mathcal{R}}$. The pair $(j_A^{\mathcal{R}}, j_G^{\mathcal{R}})$ is anti-covariant in the sense that, for all $a \in A$ and all $r \in G$,

$$j_A^{\mathcal{R}}(\alpha_r(a)) = j_G^{\mathcal{R}}(r)^{-1} j_A^{\mathcal{R}}(a) j_G^{\mathcal{R}}(r).$$

The images $j_A^{\mathcal{R}}(A)$ and $j_G^{\mathcal{R}}(G)$ are contained in $\mathcal{M}_r((A \rtimes_\alpha G)^{\mathcal{R}})$, the right centralizer algebra of $(A \rtimes_\alpha G)^{\mathcal{R}}$, so we have

$$j_A^{\mathcal{R}} : A \to \mathcal{M}_r((A \rtimes_\alpha G)^{\mathcal{R}}) \subset B((A \rtimes_\alpha G)^{\mathcal{R}}),$$
$$j_G^{\mathcal{R}} : G \to \mathcal{M}_r((A \rtimes_\alpha G)^{\mathcal{R}}) \subset B((A \rtimes_\alpha G)^{\mathcal{R}}).$$

For the operator norm in $B((A \rtimes_\alpha G)^{\mathcal{R}})$ the estimates

$$\left\| j_A^{\mathcal{R}}(a) \right\| \leq \sup_{(\pi,U)\in\mathcal{R}} \|\pi(a)\| \leq C^{\mathcal{R}} \|a\|,$$

where $a \in A$, and

$$\left\| j_G^{\mathcal{R}}(r) \right\| \leq \nu^{\mathcal{R}}(r),$$

where $r \in G$, hold.

If, in addition, A has a bounded approximate right identity, then $j_A^{\mathcal{R}}$ is non-degenerate.

Proof. The proof is similar to the proof of the corresponding statements in Proposition 2.6.4 and the details are therefore omitted. □

Proposition 2.6.6. *Let (A, G, α) be a Banach algebra dynamical system and let $\mathcal{R}$ be a non-empty uniformly bounded class of continuous covariant representations of continuous covariant representations. For $a \in A$ and $r \in G$, let $i_A^{\mathcal{R}}(a)$ and $i_G^{\mathcal{R}}(r)$ be as in Proposition 2.6.4, and let $j_A^{\mathcal{R}}(a)$ and $j_G^{\mathcal{R}}(r)$ be as in Proposition 2.6.5. Then $((i_A^{\mathcal{R}}(a), j_A^{\mathcal{R}}(a))$ and $(i_G^{\mathcal{R}}(r), j_G^{\mathcal{R}}(r))$ are both double centralizers of $(A \rtimes_\alpha G)^{\mathcal{R}}$, and we have*

$$\left\| (i_A^{\mathcal{R}}(a), j_A^{\mathcal{R}}(a)) \right\| \leq \sup_{(\pi,U)\in\mathcal{R}} \|\pi(a)\| \leq C^{\mathcal{R}} \|a\|,$$

and

$$\left\| (i_G^{\mathcal{R}}(r), j_G^{\mathcal{R}}(r)) \right\| \leq \nu^{\mathcal{R}}(r).$$

Furthermore, the maps $a \mapsto (i_A^{\mathcal{R}}(a), j_A^{\mathcal{R}}(a))$ *and* $r \mapsto (i_G^{\mathcal{R}}(r), j_G^{\mathcal{R}}(r))$ *are homomorphisms of* A *into* $\mathcal{M}((A \rtimes_\alpha G)^{\mathcal{R}})$ *and of* G *into* $\mathcal{M}((A \rtimes_\alpha G)^{\mathcal{R}})$*, respectively, and the pair* $((i_A^{\mathcal{R}}, j_A^{\mathcal{R}}), (i_G^{\mathcal{R}}, j_G^{\mathcal{R}}))$ *is covariant in the sense that*

$$(i_A^{\mathcal{R}}(\alpha_r(a)), j_A^{\mathcal{R}}(\alpha_r(a))) = (i_G^{\mathcal{R}}(r), j_G^{\mathcal{R}}(r)) \cdot (i_A^{\mathcal{R}}(a), j_A^{\mathcal{R}}(a)) \cdot (i_G^{\mathcal{R}}(r), j_G^{\mathcal{R}}(r))^{-1},$$

for all $a \in A$ *and all* $r \in G$*.*

Moreover, if (A, G, α) *and* $\mathcal{R}$ *are involutive, then*

$$(i_A^{\mathcal{R}}, j_A^{\mathcal{R}}) : A \to \mathcal{M}((A \rtimes_\alpha G)^{\mathcal{R}})$$

is an involutive homomorphism, and $(i_G^{\mathcal{R}}(r), j_G^{\mathcal{R}}(r))^* = (i_G^{\mathcal{R}}(r^{-1}), j_G^{\mathcal{R}}(r^{-1}))$*, for all* $r \in G$*.*

Proof. Let $a \in A$ and suppose $f, g \in C_c(G, A)$. Then the computation, for $s \in G$,

$$\begin{aligned} f * (i_A(a)g)(s) &= \int_G f(r)\alpha_r(i_A(a)g(r^{-1}s))\,dr \\ &= \int_G f(r)\alpha_r(a)\alpha_r(g(r^{-1}s))\,dr \\ &= (j_A(a)f) * g(s) \end{aligned}$$

shows that $(i_A(a), j_A(a))$ is a double centralizer of $C_c(G, A)$. By continuity and density, the same holds for $(i_A^{\mathcal{R}}(a), j_A^{\mathcal{R}}(a))$ and $(A \rtimes_\alpha G)^{\mathcal{R}}$. Similarly, if $r, s \in G$ and $f, g \in C_c(G, A)$, then

$$\begin{aligned} f * (i_G(r)g)(s) &= \int_G f(t)\alpha_t((i_G(r)g)(t^{-1}s))\,dt \\ &= \int_G f(t)\alpha_t(\alpha_r(g(r^{-1}t^{-1}s)))\,dt \\ &= \int_G f(t)\alpha_{tr}(g((tr)^{-1}s))\,dt \\ &= \int_G \Delta(r^{-1})f(tr^{-1})\alpha_t(g(t^{-1}s))\,dt \\ &= \int_G (j_G(r)f)(t)\alpha_t(g(t^{-1}s))\,dt \\ &= (j_G(r)f) * g(s) \end{aligned}$$

implies that $(i_G^{\mathcal{R}}(r), j_G^{\mathcal{R}}(r))$ is a double centralizer of $(A \rtimes_\alpha G)^{\mathcal{R}}$.

The fact that the maps are homomorphisms and the covariance property follow directly from the corresponding statements in Proposition 2.6.4 and Proposition 2.6.5, and the definition of the inverse and the multiplication in the double centralizer algebra.

As to the final statement, suppose that (A, G, α) is involutive, and that $\mathcal{R}$ consists of involutive representations. To show that the homomorphism (i_A, j_A) from A into

$\mathcal{M}((A \rtimes_\alpha G)^{\mathcal{R}})$ is involutive, we have to show that $(i_A^{\mathcal{R}}(a), j_A^{\mathcal{R}}(a))^* = (i_A^{\mathcal{R}}(a^*), j_A^{\mathcal{R}}(a^*))$ for all $a \in A$, i.e., that $(j_A^{\mathcal{R}}(a)^*, i_A^{\mathcal{R}}(a)^*) = (i_A^{\mathcal{R}}(a^*), j_A^{\mathcal{R}}(a^*))$. Recalling the definitions (2.3.1) and (2.6.4), we find, for $f \in C_c(G, A)$ and $s \in G$, that

$$\begin{aligned}
[j_A(a)^* f](s) &= [j_A(a) f^*]^*(s) \\
&= \Delta(s^{-1})\alpha_s\left[\left\{(j_A(a)f^*)(s^{-1})\right\}^*\right] \\
&= \Delta(s^{-1})\alpha_s\left[\left\{f^*(s^{-1})\alpha_{s^{-1}}(a)\right\}^*\right] \\
&= \Delta(s^{-1})\alpha_s\left[\left\{\Delta(s)\alpha_{s^{-1}}(f(s)^*)\alpha_{s^{-1}}(a)\right\}^*\right] \\
&= a^* f(s) \\
&= [i_A(a^*)f](s).
\end{aligned}$$

and

$$\begin{aligned}
[i_A(a)^* f](s) &= [i_A(a) f^*]^*(s) \\
&= \Delta(s^{-1})\alpha_s\left[\left\{(i_A(a)f^*)(s^{-1})\right\}^*\right] \\
&= \Delta(s^{-1})\alpha_s\left[\left\{a f^*(s^{-1})\right\}^*\right] \\
&= \Delta(s^{-1})\alpha_s\left[\left\{a\Delta(s)\alpha_{s^{-1}}(f(s)^*)\right\}^*\right] \\
&= f(s)\alpha_s(a^*) \\
&= [j_A(a^*)f](s),
\end{aligned}$$

By continuity and density, this implies that $(j_A^{\mathcal{R}}(a)^*, i_A^{\mathcal{R}}(a)^*) = (i_A^{\mathcal{R}}(a^*), j_A^{\mathcal{R}}(a^*))$, as desired.

A similar unwinding of the definitions establishes, by continuity and density, that $i_G^{\mathcal{R}}(r)^* = j_G^{\mathcal{R}}(r^{-1})$, for all $r \in G$. Taking adjoints, this implies $j_G^{\mathcal{R}}(r)^* = i_G^{\mathcal{R}}(r^{-1})$, hence $(i_G^{\mathcal{R}}(r), j_G^{\mathcal{R}}(r))^* = (j_G^{\mathcal{R}}(r)^*, (i_G^{\mathcal{R}}(r)^*) = (i_G^{\mathcal{R}}(r^{-1}), j_G^{\mathcal{R}}(r^{-1}))$, for all $r \in G$. $\square$

2.7 Representations: from $(A \rtimes_\alpha G)^{\mathcal{R}}$ to (A, G, α)

As already indicated in the previous section, Theorem 2.6.1 and Proposition 2.6.4 provide a means to generate a covariant representation of (A, G, α) from a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$, as follows. If A has a bounded approximate left identity, then the same holds for $(A \rtimes_\alpha G)^{\mathcal{R}}$, by Corollary 2.4.6, and hence any non-degenerate bounded representation T of $(A \rtimes_\alpha G)^{\mathcal{R}}$ yields a bounded representation $\overline{T}$ of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$, by Theorem 2.6.1. Since, by Proposition 2.6.4, the images $i_A^{\mathcal{R}}(A)$ and $i_G^{\mathcal{R}}(G)$ are contained in $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$, the pair of maps $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is meaningfully defined and will then be a covariant representation of (A, G, α), since the covariance requirement is automatically satisfied as a consequence of the covariance property of $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$, the latter being part of Proposition 2.6.4. Some natural questions that arise are, e.g., whether this covariant representation is continuous, and, if so, whether it is $\mathcal{R}$-continuous. We will now investigate these and related matters, and incorporate some of the results from Section 2.5 (the passage in the other direction, from $\mathcal{R}$-continuous covariant representations of (A, G, α) to bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$) in the process. After that, the proofs of our main results in Section 2.8 will be a mere formality.

Recall from Definition 2.4.5 and Corollary 2.4.6 that $M_l^{\mathcal{R}}$ denotes the infimum of the upper bounds of the approximate left identities of $(A \rtimes_\alpha G)^{\mathcal{R}}$, with estimate $M_l^{\mathcal{R}} \leq C^{\mathcal{R}} M_l^A N^{\mathcal{R}}$, where M_l^A denotes the infimum of the upper bounds of the approximate left identities of A.

Proposition 2.7.1. *Let (A, G, α) be a Banach algebra dynamical system, where A has a bounded approximate left identity, and let $\mathcal{R}$ be a non-empty uniformly bounded class of continuous covariant representations. Let $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ be the continuous covariant representation of (A, G, α) on $(A \rtimes_\alpha G)^{\mathcal{R}}$, as in Proposition 2.6.4.*

Suppose that T is a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ in a Banach space X, and let $\overline{T}$ be the associated bounded representation of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ in X, as in Theorem 2.6.1. Then the pair $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is a non-degenerate continuous covariant representation of (A, G, α) in X. For the operator norm on the bounded operators on the representation space the estimates

$$\left\| \left(\overline{T} \circ i_A^{\mathcal{R}}\right)(a) \right\| \leq M_l^{\mathcal{R}} \left\| T \right\| \sup_{(\pi, U) \in \mathcal{R}} \left\| \pi(a) \right\| \leq M_l^{\mathcal{R}} \left\| T \right\| C^{\mathcal{R}} \left\| a \right\|,$$

where $a \in A$, and

$$\left\| \left(\overline{T} \circ i_G^{\mathcal{R}}\right)(r) \right\| \leq M_l^{\mathcal{R}} \left\| T \right\| \nu^{\mathcal{R}}(r),$$

where $r \in G$, hold.

If a closed subspace of X is invariant for T, it is invariant for $\overline{T} \circ i_A^{\mathcal{R}}$ and $\overline{T} \circ i_G^{\mathcal{R}}$, and if Y is a Banach space, $S : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(Y)$ a representation and $\Phi \in B(X, Y)$ intertwines T and S, then Φ intertwines $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ and $(\overline{S} \circ i_A^{\mathcal{R}}, \overline{S} \circ i_G^{\mathcal{R}})$.

If, in addition, (A, G, α), $\mathcal{R}$, and T are involutive, then $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is involutive.

Moreover, if, in the not necessarily involutive case, (π, U) is an non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α), with corresponding non-degenerate bounded representation $(\pi \rtimes U)^{\mathcal{R}}$ of $(A \rtimes_\alpha G)^{\mathcal{R}}$, then

$$\left(\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_A^{\mathcal{R}}, \overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_G^{\mathcal{R}} \right) = (\pi, U). \tag{2.7.1}$$

Although it does not follow from the estimates for the operator norm in the theorem, if all elements of $\mathcal{R}$ are non-degenerate, then it is (in analogy with the continuous covariant representation $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ of (A, G, α) in $(A \rtimes_\alpha G)^{\mathcal{R}}$), actually true that $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is $\mathcal{R}$-continuous, see Theorem 2.7.3.

Note that the final statement of the theorem implies the injectivity of the assignment $(\pi, U) \to (\pi \rtimes U)^{\mathcal{R}}$ on the non-degenerate $\mathcal{R}$-continuous covariant representations if A has a bounded approximate left identity, as was already announced following Definition 2.5.1.

Proof. Let T be a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ in the Banach space X. As already remarked preceding the theorem, the definitions

$$\pi := \overline{T} \circ i_A^{\mathcal{R}} \quad \text{and} \quad U := \overline{T} \circ i_G^{\mathcal{R}}$$

are meaningful and provide a covariant representation (π, U) of (A, G, α). We show that it has the properties as claimed, and start with the bounds for $\|\pi\|$ and $\|U_r\|$, for $r \in G$. Let $\varepsilon > 0$, then $(A \rtimes_\alpha G)^{\mathcal{R}}$ has an $(M_l^{\mathcal{R}} + \varepsilon)$-bounded approximate left identity. Since Theorem 2.6.1 and Proposition 2.6.4 provide a bound for $\|\overline{T}\|$, $\|i_A^{\mathcal{R}}(a)\|$, and $\|i_G^{\mathcal{R}}(r)\|$, we have, for $a \in A$,

$$\begin{aligned} \|\pi(a)\| &\leq \|\overline{T}\| \, \|i_A^{\mathcal{R}}(a)\| \\ &\leq (M_l^{\mathcal{R}} + \varepsilon) \, \|T\| \sup_{(\rho, V) \in \mathcal{R}} \|\rho(a)\| \\ &\leq (M_l^{\mathcal{R}} + \varepsilon) \, \|T\| \, C^{\mathcal{R}} \, \|a\| \end{aligned}$$

and, for $r \in G$,

$$\|U_r\| \leq \|\overline{T}\| \, \|i_G^{\mathcal{R}}(r)\| \leq (M_l^{\mathcal{R}} + \varepsilon) \, \|T\| \, \nu^{\mathcal{R}}(r).$$

Since $\varepsilon > 0$ was arbitrary, this establishes the estimates in the theorem.

We have to prove that π is non-degenerate and that U is strongly continuous. Starting with π, by Remark 2.2.8 it has to be shown that $\pi(u_i)x \to x$ for all $x \in X$, where (u_i) is a bounded approximate left identity of A. By the boundedness of π, which we already established, and the boundedness of (u_i), it is sufficient to establish this for x in a dense subset of X. Now since T is non-degenerate and $q^{\mathcal{R}}(C_c(G) \otimes A)$ is dense in $(A \rtimes_\alpha G)^{\mathcal{R}}$ by Corollary 2.3.6, $T(q^{\mathcal{R}}(C_c(G) \otimes A)) \cdot X$ is dense in X. So let $x \in X$ and $f \in C_c(G) \otimes A$, then by (2.6.2) in Theorem 2.6.1,

$$\begin{aligned} \pi(u_i) T(q^{\mathcal{R}}(f))x &= \overline{T}(i_A^{\mathcal{R}}(u_i)) T(q^{\mathcal{R}}(f))x \\ &= T[i_A^{\mathcal{R}}(u_i)(q^{\mathcal{R}}(f))]x \\ &= T[q^{\mathcal{R}}(i_A(u_i)f)]y \\ &\to T(q^{\mathcal{R}}(f))x, \end{aligned}$$

where the last step is by Lemma 2.5.4, Lemma 2.3.5 and the boundedness of T.

Now we turn to the strong continuity of U, Since we have already established that $\|U_r\| \leq \nu^{\mathcal{R}}(r)$, for $r \in G$, and $\nu^{\mathcal{R}}$ is bounded on compact sets, Corollary 2.2.5 shows that it is sufficient to show strong continuity of U in e when acting on a dense subset of X. For this set we choose $T((A \rtimes_\alpha G)^{\mathcal{R}}) \cdot X$, which is dense by the non-degeneracy of T, and then by linearity it is sufficient to show strong continuity in e when acting on elements of the form $T(c)y$, where $c \in (A \rtimes_\alpha G)^{\mathcal{R}}$, and $y \in X$. So let $x = T(c)y \in X$, and let $r_i \to e$. Then by (2.6.2) in Theorem 2.6.1 we find

$$U_{r_i} x = \overline{T}(i_G^{\mathcal{R}}(r_i)) T(c)y = T(i_G^{\mathcal{R}}(r_i)(c))y.$$

By Proposition 2.6.4, $i_G^{\mathcal{R}}$ is strongly continuous. Hence, by the continuity of T,

$$U_{r_i}x = T(i_G^{\mathcal{R}}(r_i)(c))y \to T(c)y = x,$$

as required.

Suppose Y is a closed invariant subspace of X for T. By Theorem 2.6.1 $\overline{T}(L)y \in Y$ for all $y \in Y$ and $L \in \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$. Applying this with $L = i_A(a)$ and $L = i_G(r)$, for $a \in A$ and $r \in G$, shows that Y is invariant for $\overline{T} \circ i_A$ and $\overline{T} \circ i_G$.

If Y is a Banach space, S a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ in Y and Φ a bounded intertwining operator for T and S, then it follows from Theorem 2.6.1 that $\Phi \circ \overline{T}(L) = \overline{S}(L) \circ \Phi$ for all $L \in \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$. Again applying this with $L = i_A(a)$ and $L = i_G(r)$, for $a \in A$ and $r \in G$, shows that $\Phi \circ [\overline{T} \circ i_A](a) = [\overline{S} \circ i_A](a) \circ \Phi$ and $\Phi \circ [\overline{T} \circ i_G](r) = [\overline{S} \circ i_G](r) \circ \Phi$.

Considering the statement on involutions, suppose that, in addition, (A, G, α) and $\mathcal{R}$ are both involutive. Let T be an involutive representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$. By Proposition 2.6.6 the homomorphism $(i_A^{\mathcal{R}}, j_A^{\mathcal{R}}) : A \to \mathcal{M}((A \rtimes_\alpha G)^{\mathcal{R}})$ is involutive and Theorem 2.6.1 shows that $\overline{T} \circ \phi_l$ is involutive. Combining these, we obtain that

$$\pi = \overline{T} \circ i_A^{\mathcal{R}} = \overline{T} \circ \left[\phi_l \circ (i_A^{\mathcal{R}}, j_A^{\mathcal{R}})\right] = \left[\overline{T} \circ \phi_l\right] \circ (i_A^{\mathcal{R}}, j_A^{\mathcal{R}})$$

is an involutive representation of A. Finally, if $r \in G$, then using the involutive property of $\overline{T} \circ \phi_l$ again, as well as Proposition 2.6.6, we see that

$$\begin{aligned}
U_r^* &= \left[\overline{T}(i_G^{\mathcal{R}}(r))\right]^* \\
&= \left[\left(\overline{T} \circ \phi_l\right)\left((i_G^{\mathcal{R}}(r), j_G^{\mathcal{R}}(r))\right)\right]^* \\
&= \left(\overline{T} \circ \phi_l\right)\left[(i_G^{\mathcal{R}}(r), j_G^{\mathcal{R}}(r))^*\right] \\
&= \left(\overline{T} \circ \phi_l\right)\left[(j_G^{\mathcal{R}}(r)^*, i_G^{\mathcal{R}}(r)^*)\right] \\
&= \left(\overline{T} \circ \phi_l\right)\left[(i_G^{\mathcal{R}}(r^{-1}), j_G^{\mathcal{R}}(r^{-1}))\right] \\
&= \overline{T}(i_G^{\mathcal{R}}(r^{-1})) = U_{r^{-1}} = U_r^{-1}.
\end{aligned}$$

Hence U is a unitary representation of G, and this completes the proof that the pair $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is involutive.

To conclude with, we consider the final statement on the recovery of an $\mathcal{R}$-continuous covariant representation (π, U) from $(\pi \rtimes U)^{\mathcal{R}}$. Starting with π, let $a \in A$. Then the compatibility equation (2.6.2) in Theorem 2.6.1, when applied with L replaced with $i_A^{\mathcal{R}}(a)$ and a replaced with $q^{\mathcal{R}}(f)$, for $f \in C_c(G, A)$, yields

$$\begin{aligned}
\overline{(\pi \rtimes U)^{\mathcal{R}}}(i_A^{\mathcal{R}}(a)) \circ (\pi \rtimes U)^{\mathcal{R}}(q^{\mathcal{R}}(f)) &= (\pi \rtimes U)^{\mathcal{R}}(i_A^{\mathcal{R}}(a)q^{\mathcal{R}}(f)) \\
&= \pi \rtimes U(i_A(a)f). \qquad (2.7.2)
\end{aligned}$$

Take an element $x \in X$ of the form $x = \pi \rtimes U(f)y$, with $f \in C_c(G, A)$. Using (2.5.3), we find that

$$\pi(a)x = \pi(a) \circ \pi \rtimes U(f)y = \pi \rtimes U(i_A(a)f)y.$$

Combining this with both sides of (2.7.2) acting on y, we see that, for such x,

$$\overline{(\pi \rtimes U)^{\mathcal{R}}}(i_A^{\mathcal{R}}(a))x = \pi(a)x. \tag{2.7.3}$$

By Proposition 2.5.5 the linear span of elements of the form $\pi \rtimes U(f)y$, with $f \in C_c(G, A)$ and $y \in X$, is dense in X, and therefore (2.7.3) implies that $\pi(a)$ and $\overline{(\pi \rtimes U)^{\mathcal{R}}}(i_A^{\mathcal{R}}(a))$ are equal.

The proof that $\overline{(\pi \rtimes U)^{\mathcal{R}}}(i_G(r)) = U_r$, for $r \in G$, is similar. □

The reconstruction formula (2.7.1) will enable us to complete our results on the continuous covariant representation $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ from Proposition 2.6.4, under the extra conditions that A has a bounded approximate left identity and that all elements of $\mathcal{R}$ are non-degenerate.

Theorem 2.7.2. *Let (A, G, α) be a Banach algebra dynamical system, where A has a bounded approximate left identity, and let $\mathcal{R}$ be a non-empty uniformly bounded class of non-degenerate continuous covariant representations. Then the non-degenerate continuous covariant representation $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ of (A, G, α) on $(A \rtimes_\alpha G)^{\mathcal{R}}$ from Proposition 2.6.4 is $\mathcal{R}$-continuous, and the associated non-degenerate bounded representation $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}}$ of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on itself coincides with the left regular representation, and is therefore contractive.*

Proof. We start by showing that $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ is $\mathcal{R}$-continuous. If $f \in C_c(G, A)$, then $i_A(f(s))i_G(s)$ is a left centralizer for all $s \in G$, and hence commutes with all right multiplications. By (2.2.4) these right multiplications can be pulled through the integral, therefore $i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}}(f) = \int_G i_A(f(s))i_G(s)\,ds$ commutes with all right multiplications as well, hence it is a left centralizer.

Let λ denote the left regular representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$. Then using (2.7.1) in the fourth step, we find that, for all $(\pi, U) \in \mathcal{R}$,

$$\begin{aligned}
\overline{(\pi \rtimes U)^{\mathcal{R}}}\left(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}}(f)\right) &= \overline{(\pi \rtimes U)^{\mathcal{R}}}\left(\int_G i_A^{\mathcal{R}}(f(s))i_G^{\mathcal{R}}(s)\,ds\right) \\
&= \int_G \overline{(\pi \rtimes U)^{\mathcal{R}}}(i_A^{\mathcal{R}}(f(s))i_G^{\mathcal{R}}(s))\,ds \\
&= \int_G \overline{(\pi \rtimes U)^{\mathcal{R}}}(i_A^{\mathcal{R}}(f(s))) \cdot \overline{(\pi \rtimes U)^{\mathcal{R}}}(i_G^{\mathcal{R}}(s))\,ds \\
&= \int_G \pi(f(s))U_s\,ds \\
&= \pi \rtimes U(f) \\
&= (\pi \rtimes U)^{\mathcal{R}}(q^{\mathcal{R}}(f)) \\
&= \overline{(\pi \rtimes U)^{\mathcal{R}}}(\lambda(q^{\mathcal{R}}(f))),
\end{aligned}$$

where diagram (2.6.1) was used in the final step. By Proposition 2.6.2 the representations $\overline{(\pi \rtimes U)^{\mathcal{R}}}$ separate the points, and it follows that

$$i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}}(f) = \lambda(q^{\mathcal{R}}(f)).$$

Consequently, $\left\|i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}}(f)\right\| \leq \|\lambda\| \left\|q^{\mathcal{R}}(f)\right\|^{\mathcal{R}} = \|\lambda\| \sigma^{\mathcal{R}}(f)$. We conclude that $i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}}$ is $\mathcal{R}$-continuous.

Next we consider the statement that $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}}$, the representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on itself, which we now know to be defined as a consequence of the first part of the proof, is the left regular representation. Let $f \in C_c(G, A)$. Then, for all $(\pi, U) \in \mathcal{R}$, the above computation shows that

$$\overline{(\pi \rtimes U)^{\mathcal{R}}}\left((i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}}(q^{\mathcal{R}}(f))\right) = \overline{(\pi \rtimes U)^{\mathcal{R}}}\left(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}}(f)\right) = \overline{(\pi \rtimes U)^{\mathcal{R}}}(\lambda(q^{\mathcal{R}}(f))),$$

so again by the point-separating property of the representations $\overline{(\pi \rtimes U)^{\mathcal{R}}}$ it follows that $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}}(q^{\mathcal{R}}(f)) = \lambda(q^{\mathcal{R}}(f))$. By continuity and density, the statement follows. $\square$

In turn, Theorem 2.7.2 enables us to understand that, as already remarked after Proposition 2.7.1, the non-degenerate continuous covariant representation obtained in that proposition is actually $\mathcal{R}$-continuous, under the extra condition that all elements of $\mathcal{R}$ are non-degenerate. In that case, there is an associated bounded representation of the crossed product again, and the following result, in which some other main results of this section have been included again for future reference, shows additionally that this two-step process is the identity.

Theorem 2.7.3. *Let (A, G, α) be a Banach algebra dynamical system, where A has a bounded approximate left identity, and let $\mathcal{R}$ be a non-empty uniformly bounded class of non-degenerate continuous covariant representations. Let $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ be the non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on $(A \rtimes_\alpha G)^{\mathcal{R}}$, as in Proposition 2.6.4 and Theorem 2.7.2.*

Suppose that T is a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ in a Banach space X, and let $\overline{T}$ be the associated representation of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ in X, as in Theorem 2.6.1. Then the pair $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) in X, and the corresponding non-degenerate bounded representation $\left((\overline{T} \circ i_A^{\mathcal{R}}) \rtimes (\overline{T} \circ i_G^{\mathcal{R}})\right)^{\mathcal{R}}$ of $(A \rtimes_\alpha G)^{\mathcal{R}}$ in X coincides with T. In particular, $\left\|(\overline{T} \circ i_A^{\mathcal{R}}) \rtimes (\overline{T} \circ i_G^{\mathcal{R}})\right\|^{\mathcal{R}} = \|T\|$.

If a closed subspace of X is invariant for T, it is invariant for $\overline{T} \circ i_A$ and $\overline{T} \circ i_G$, and if Y is a Banach space, $S : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(Y)$ a representation and $\Phi \in B(X, Y)$ intertwines T and S, then Φ intertwines $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ and $(\overline{S} \circ i_A^{\mathcal{R}}, \overline{S} \circ i_G^{\mathcal{R}})$.

If, in addition, (A, G, α), $\mathcal{R}$, and T are involutive, then $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is involutive.

Proof. The statements concerning invariant subspaces, intertwiners and involutions have already been proven in Proposition 2.7.1.

Denote $\pi := \overline{T} \circ i_A^{\mathcal{R}}$ and $U := \overline{T} \circ i_G^{\mathcal{R}}$. Proposition 2.7.1 asserts that (π, U) is a continuous covariant representation, hence its integrated form $\pi \rtimes U : C_c(G, A) \to B(X)$ is defined. We claim that, for all $f \in C_c(G, A)$,

$$\pi \rtimes U(f) = T(q^{\mathcal{R}}(f)). \tag{2.7.4}$$

Assuming this for the moment, we see that, for all $f \in C_c(G, A)$,

$$\|\pi \rtimes U(f)\| = \left\|T(q^{\mathcal{R}}(f))\right\| \leq \|T\| \left\|q^{\mathcal{R}}(f)\right\|^{\mathcal{R}} = \|T\| \, \sigma^{\mathcal{R}}(f).$$

Hence (π, U) is $\mathcal{R}$-continuous, and consequently the corresponding bounded representation $(\pi \rtimes U)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(X)$ can indeed be defined and we conclude, using the definition and (2.7.4), that $(\pi \rtimes U)^{\mathcal{R}}(q^{\mathcal{R}}(f)) = \pi \rtimes U(f) = T(q^{\mathcal{R}}(f))$, for all $f \in C_c(G, A)$. By the density of $q^{\mathcal{R}}(C_c(G, A))$ in $(A \rtimes_\alpha G)^{\mathcal{R}}$, this implies that $(\pi \rtimes U)^{\mathcal{R}} = T$. The statement concerning the norms then follows from Theorem 2.5.6.

Hence it remains to establish (2.7.4). For this, let f, $g \in C_c(G, A)$. Then (2.6.2) implies that

$$\begin{aligned} U_s \circ T(q^{\mathcal{R}}(g)) &= \overline{T}(i_G^{\mathcal{R}}(s)) \circ T(q^{\mathcal{R}}(g)) \\ &= T(i_G^{\mathcal{R}}(s)(q^{\mathcal{R}}(g))) \\ &= T\left(q^{\mathcal{R}}(i_G(s)g)\right). \end{aligned}$$

Similarly, we have, for all $a \in A$, and $h \in C_c(G, A)$,

$$\pi(a) \circ T(q^{\mathcal{R}}(h)) = T\left(q^{\mathcal{R}}(i_A(a)h)\right) x.$$

Combining these, we find that, for $s \in G$,

$$\begin{aligned} \pi(f(s))U_s \circ T(q^{\mathcal{R}}(g)) &= \pi(f(s)) \circ T\left(q^{\mathcal{R}}(i_G(s)g)\right) \\ &= T\left(q^{\mathcal{R}}(i_A(f(s))i_G(s)g)\right) \\ &= T\left(i_A^{\mathcal{R}}(f(s))i_G^{\mathcal{R}}(s)q^{\mathcal{R}}(g)\right). \end{aligned}$$

We conclude that, for all f, $g \in C_c(G, A)$,

$$\pi \rtimes U(f) \circ T(q^{\mathcal{R}}(g)) = \int_G T\left(i_A^{\mathcal{R}}(f(s))i_G^{\mathcal{R}}(s)q^{\mathcal{R}}(g)\right) ds. \tag{2.7.5}$$

On the other hand, with λ denoting the left regular representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$, Theorem 2.7.2 implies that, for f, $g \in C_c(G, A)$,

$$\begin{aligned} q^{\mathcal{R}}(f) * q^{\mathcal{R}}(g) &= \lambda(q^{\mathcal{R}}(f))q^{\mathcal{R}}(g) \\ &= (i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}}(q^{\mathcal{R}}(f))q^{\mathcal{R}}(g) \\ &= i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}}(f)q^{\mathcal{R}}(g) \\ &= \int_G i_A^{\mathcal{R}}(f(s))i_G^{\mathcal{R}}(s)q^{\mathcal{R}}(g) \, ds. \end{aligned}$$

Applying the bounded homomorphism T to this relation yields

$$T(q^{\mathcal{R}}(f)) \circ T(q^{\mathcal{R}}(g)) = \int_G T\left(i_A^{\mathcal{R}}(f(s))i_G^{\mathcal{R}}(s)q^{\mathcal{R}}(g)\right) ds, \tag{2.7.6}$$

for all f, $g \in C_c(G, A)$. For $x \in X$, comparing (2.7.5) and (2.7.6) and applying them to x, we see that

$$\pi \rtimes U(f)\left(T(q^{\mathcal{R}}(g))x\right) = T(q^{\mathcal{R}}(f))\left(T(q^{\mathcal{R}}(g))x\right). \tag{2.7.7}$$

Now since T is a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$, the restriction of T to the dense subalgebra $q^{\mathcal{R}}(C_c(G, A)$ must be non-degenerate as well, and so elements of the form $T(q^{\mathcal{R}}(g))x$ are dense in X. Hence (2.7.7) implies that $\pi \rtimes U(f) = T(q^{\mathcal{R}}(f))$ holds for all $f \in C_c(G, A)$, as desired. $\square$

2.8 Representations: general correspondence

In this section, which can be viewed as the conclusion of the analysis in the preceding parts of this paper, we put the pieces together without too much extra effort. We give references to the relevant definitions, in order to enhance accessibility of the results to the reader who is not familiar with the details of the Sections 2.2 through 2.7. Section 2.9 contains some applications.

As an introductory remark for the reader who is familiar with the preceding sections, we note that Theorem 2.5.6 describes the properties of the passage from $\mathcal{R}$-continuous covariant representations of (A, G, α) to bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$. Such a passage is always possible, without further assumptions on the Banach algebra dynamical system or the covariant representations. Proposition 2.7.1, valid under the condition that A has a bounded approximate left identity, goes in the opposite direction, but it is only for non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ that a (non-degenerate) continuous covariant representation of (A, G, α) is constructed. If one starts with an non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α), passes to the associated non-degenerated bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$, and then goes back to (A, G, α) again, the same Proposition 2.7.1 shows that one retrieves the original covariant representation of (A, G, α). If, in addition, all elements of $\mathcal{R}$ are themselves non-degenerate, then Proposition 2.7.1 can be improved to Theorem 2.7.3, where it is concluded that the (non-degenerate) continuous covariant representation of (A, G, α) as constructed from a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ is actually $\mathcal{R}$-continuous. Hence it is possible to go in the first direction again, thus obtaining a bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$, and, according to the same Theorem 2.7.3, this is the representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ one started with. As it turns out, if we impose these conditions on A (having a bounded approximate left identity) and $\mathcal{R}$ (consisting of non-degenerate continuous covariant representations), and restrict our attention to *non-degenerate* $\mathcal{R}$-continuous covariant representations of (A, G, α) and *non-degenerate* bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$, then we obtain a bijection, according to our main general result, the general correspondence in Theorem 2.8.1 below.

We now turn to the formulation of the result, recalling the relevant notions and definitions as a preparation, and introducing two new notations for (covariant) representations of a certain type. If (A, G, α) is a Banach algebra dynamical system

(Definition 2.2.10), $\mathcal{R}$ is a non-empty uniformly bounded (Definition 2.3.1) class of non-degenerate continuous covariant representations (Definition 2.2.12) of (A, G, α), and $\mathcal{X}$ is a non-empty class of Banach spaces, we let $\mathrm{Covrep}_{\mathrm{nd,c}}^{\mathcal{R}}((A, G, \alpha), \mathcal{X})$ denote the non-degenerate $\mathcal{R}$-continuous (Definitions 2.3.2 and 2.5.1) representations of (A, G, α) in spaces from $\mathcal{X}$, and we let $\mathrm{Rep}_{\mathrm{nd,b}}((A \rtimes_\alpha G)^{\mathcal{R}}, \mathcal{X})$ denote the non-degenerate bounded representations of the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ (Definition 2.3.2) in spaces from $\mathcal{X}$. There need not be a relation between the representation spaces corresponding to the elements of $\mathcal{R}$ and the spaces in $\mathcal{X}$.

Furthermore, we let $\mathcal{I}^{\mathcal{R}}$ denote the assignment $(\pi, U) \to (\pi \rtimes U)^{\mathcal{R}}$, sending an $\mathcal{R}$-continuous covariant representation of (A, G, α) to a bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$, as explained following Remark 2.5.2. If A has a bounded approximate left identity, then we let $\mathcal{S}^{\mathcal{R}}$ denote the assignment $T \to (\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$, as in Proposition 2.7.1 and Theorem 2.7.3, sending a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ to a non-degenerate continuous covariant representation of (A, G, α), obtained by first constructing a non-degenerate bounded representation $\overline{T}$ of the left centralizer algebra of $(A \rtimes_\alpha G)^{\mathcal{R}}$, compatible with T, and subsequently composing this with the canonical continuous covariant representation $(i_A^{\mathcal{R}}, i_A^{\mathcal{R}})$ of (A, G, α) in $(A \rtimes_\alpha G)^{\mathcal{R}}$, the images of which actually lie in this left centralizer algebra (see Proposition 2.6.4).

The notations $\mathcal{I}^{\mathcal{R}}$ and $\mathcal{S}^{\mathcal{R}}$ are meant to suggest "integration" and "separation", respectively.

Finally, we recall the notions of an involutive Banach algebra dynamical system (Definition 2.2.10), of an involutive representation of such a system (Definition 2.2.12), and of bounded intertwining operators between (covariant) representations (final part of Section 2.2.3).

Theorem 2.8.1 (General correspondence theorem)**.** *Let (A, G, α) be a Banach algebra dynamical system, where A has a bounded approximate left identity, let $\mathcal{R}$ be a non-empty uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α), and let $\mathcal{X}$ be a non-empty class of Banach spaces. Then the restriction of $\mathcal{I}^{\mathcal{R}}$ yields a bijection*

$$\mathcal{I}^{\mathcal{R}} : \mathrm{Covrep}_{\mathrm{nd,c}}^{\mathcal{R}}((A, G, \alpha), \mathcal{X}) \to \mathrm{Rep}_{\mathrm{nd,b}}((A \rtimes_\alpha G)^{\mathcal{R}}, \mathcal{X}),$$

and the restriction of $\mathcal{S}^{\mathcal{R}}$ yields a bijection

$$\mathcal{S}^{\mathcal{R}} : \mathrm{Rep}_{\mathrm{nd,b}}((A \rtimes_\alpha G)^{\mathcal{R}}, \mathcal{X}) \to \mathrm{Covrep}_{\mathrm{nd,c}}^{\mathcal{R}}((A, G, \alpha), \mathcal{X}),$$

In fact, these restrictions of $\mathcal{I}^{\mathcal{R}}$ and $\mathcal{S}^{\mathcal{R}}$ are inverse to each other.

Furthermore, both these restrictions of $\mathcal{I}^{\mathcal{R}}$ and $\mathcal{S}^{\mathcal{R}}$ preserve the set of closed invariant subspaces for an element of their domain, as well as the Banach space of bounded intertwining operators between two elements of their domain.

If (A, G, α) and $\mathcal{R}$ are involutive, then both these restrictions of $\mathcal{I}^{\mathcal{R}}$ and $\mathcal{S}^{\mathcal{R}}$ preserve the property of being involutive.

Proof. According to Theorem 2.5.6 and Theorem 2.7.3, and also taking into account that $\mathcal{I}^{\mathcal{R}}$ and $\mathcal{S}^{\mathcal{R}}$ obviously preserve the representation space, these restricted maps are indeed meaningfully defined with domains and codomains as in the statement. Proposition 2.7.1 shows that $\mathcal{S}^{\mathcal{R}}(\mathcal{I}^{\mathcal{R}}((\pi,U))) = (\pi,U)$, for each non-degenerate $\mathcal{R}$-continuous covariant representation (π,U) of (A,G,α), whereas Theorem 2.7.3 asserts that $\mathcal{I}^{\mathcal{R}}(\mathcal{S}^{\mathcal{R}}(T)) = T$, for each non-degenerate bounded representation T of $(A \rtimes_\alpha G)^{\mathcal{R}}$. This settles the bijectivity statements.

Theorem 2.5.6 and Theorem 2.7.3 contain the statements about preservation of closed invariant subspaces, intertwining operators and the property of being involutive.

□

Remark 2.8.2. The map $\mathcal{S}^{\mathcal{R}}$, associated with a Banach algebra dynamical system (A,G,α), where A has a bounded left approximate identity, and a non-empty uniformly bounded class $\mathcal{R}$ of non-degenerate continuous covariant representations of (A,G,α), can be made explicit by recalling how Theorem 2.6.1 was used in its definition. Indeed, let T be a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ in a Banach space X. We recall the bounded approximate left identity $(q^{\mathcal{R}}(z_V \otimes u_i))$ of $(A \rtimes_\alpha G)^{\mathcal{R}}$ of Theorem 2.4.4; here (u_i) is a bounded approximate left identity of A, V runs through a neighbourhood basis $\mathcal{Z}$ of $e \in G$, of which all elements are contained in a fixed compact subset of G, and $z_V \in C_c(G)$ is positive, with total integral equal to 1, and supported in V. Let $x \in X$. Then by (2.6.3) we find, for $a \in A$,

$$\begin{aligned}(\overline{T} \circ i_A^{\mathcal{R}})(a)x &= \overline{T}(i_A^{\mathcal{R}}(a))x \\ &= \lim_{(V,i)} T\left[i_A^{\mathcal{R}}(a)\left(q^{\mathcal{R}}(z_V \otimes u_i)\right)\right]x \\ &= \lim_{(V,i)} T\left[q^{\mathcal{R}}(z_V \otimes au_i)\right]x,\end{aligned} \tag{2.8.1}$$

and, for $r \in G$,

$$\begin{aligned}(\overline{T} \circ i_G^{\mathcal{R}})(r)x &= \overline{T}(i_G^{\mathcal{R}}(r))x \\ &= \lim_{(V,i)} T\left[i_G^{\mathcal{R}}(r)\left(q^{\mathcal{R}}(z_V \otimes u_i)\right)\right]x \\ &= \lim_{(V,i)} T\left[q^{\mathcal{R}}(z_V(r^{-1}\cdot) \otimes \alpha_r(u_i))\right]x.\end{aligned} \tag{2.8.2}$$

Denoting s-lim for the limit in the strong operator topology, it follows that

$$\mathcal{S}^{\mathcal{R}}(T) = \left(a \mapsto \text{s-}\lim_{(V,i)} T\left[q^{\mathcal{R}}(z_V \otimes au_i)\right], r \mapsto \text{s-}\lim_{(V,i)} T\left[q^{\mathcal{R}}(z_V(r^{-1}\cdot) \otimes \alpha_r(u_i))\right]\right).$$

Remark 2.8.3. Any non-degenerate continuous covariant representation (π,U) of (A,G,α) in a Banach space X is an element of $\mathrm{Covrep}_{\mathrm{nd,c}}^{\mathcal{R}}((A,G,\alpha),\mathcal{X})$ with $\mathcal{R} = \{(\pi,U)\}$ and $\mathcal{X} = \{X\}$. If A has a bounded approximate left identity, then, for $a \in A$, $r \in G$ and $x \in X$, inserting (2.8.1) and (2.8.2) into $\mathcal{S}^{\mathcal{R}}(\mathcal{I}^{\mathcal{R}}((\pi,U))) = (\pi,U)$, as known from Theorem 2.8.1, yields, with the $z_V \otimes u_i$ as in Remark 2.8.2,

$$\begin{aligned}
\pi(a)x &= \left(\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_A^{\mathcal{R}}\right)(a)x \\
&= \lim_{(V,i)} (\pi \rtimes U)^{\mathcal{R}} \left[q^{\mathcal{R}}(z_V \otimes a u_i)\right] x \\
&= \lim_{(V,i)} \int_G z_V(s)\pi(a u_i) U_s x \, ds, \\
U_r x &= \left(\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_G^{\mathcal{R}}\right)(r)x \\
&= \lim_{(V,i)} (\pi \rtimes U)^{\mathcal{R}} \left[q^{\mathcal{R}}(z_V(r^{-1}\cdot) \otimes \alpha_r(u_i))\right] x \\
&= \lim_{(V,i)} \int_G z_V(r^{-1}s)\pi(\alpha_r(u_i)) U_s x \, ds.
\end{aligned}$$

These formulas, valid for an arbitrary non-degenerate continuous covariant representation (π, U) of (A, G, α), where A has a bounded approximate left identity, can also be obtained more directly, by writing $x = \lim_{(V,i)} \pi \rtimes U(z_V \otimes u_i)x$ (using Remark 2.2.8) and then using (2.5.3) in Proposition 2.5.3.

Remark 2.8.4. One also has norm estimates related to the maps $\mathcal{I}^{\mathcal{R}}$ and $\mathcal{S}^{\mathcal{R}}$.

As to $\mathcal{I}^{\mathcal{R}}$, if we assume that (A, G, α) is a Banach algebra dynamical system, and that $\mathcal{R}$ is a non-empty uniformly bounded class of continuous covariant representations of (A, G, α), then, if (π, U) is an $\mathcal{R}$-continuous covariant representation of (A, G, α), (2.3.3) yields that

$$\left\|\mathcal{I}^{\mathcal{R}}((\pi, U))q^{\mathcal{R}}(f)\right\| = \|\pi \rtimes U(f)\| \leq \|\pi\| \, \|f\|_{L^1(G,A)} \sup_{s \in \operatorname{supp}(f)} \|U_s\|, \tag{2.8.3}$$

for all $f \in C_c(G, A)$.

In order to give estimates for $\mathcal{S}^{\mathcal{R}}$, we assume that (A, G, α) is a Banach algebra dynamical system, with A having an approximate left identity, and that $\mathcal{R}$ is a non-empty uniformly bounded class of continuous covariant representations of (A, G, α). We recall the relevant constants

$$C^{\mathcal{R}} = \sup_{(\pi,U) \in \mathcal{R}} \|\pi\|, \quad \nu^{\mathcal{R}}(r) = \sup_{(\pi,U) \in \mathcal{R}} \|U_r\|, \quad N^{\mathcal{R}} = \inf_{V \in \mathcal{Z}} \sup_{r \in V} \nu^{\mathcal{R}}(r),$$

where $\mathcal{Z}$ is a neighbourhood basis of $e \in G$ which is contained in a fixed compact set (see Definition 2.4.3 and the subsequent paragraph, showing that $N^{\mathcal{R}}$ does not depend on the choice of such $\mathcal{Z}$). Furthermore, if $\mathcal{A}$ is a normed algebra with a bounded approximate left identity, then we recall that $M_l^{\mathcal{A}}$ denotes the infimum of the upper bounds of the approximate left identities of $\mathcal{A}$, and that we write $M_l^{\mathcal{R}}$ for $M_l^{(A \rtimes_\alpha G)^{\mathcal{R}}}$. Then, if T is a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$, Proposition 2.7.1 shows that, for $a \in A$,

$$\left\|\left(\overline{T} \circ i_A^{\mathcal{R}}\right)(a)\right\| \leq M_l^{\mathcal{R}} \|T\| \sup_{(\pi,U) \in \mathcal{R}} \|\pi(a)\| \leq M_l^{\mathcal{R}} \|T\| \, C^{\mathcal{R}} \|a\| .$$

In particular,

$$\left\|\overline{T} \circ i_A^{\mathcal{R}}\right\| \leq M_l^{\mathcal{R}} C^{\mathcal{R}} \|T\|. \tag{2.8.4}$$

Proposition 2.7.1 also yields that, for $r \in G$,

$$\left\|\left(\overline{T} \circ i_G^{\mathcal{R}}\right)(r)\right\| \leq M_l^{\mathcal{R}} \|T\| \nu^{\mathcal{R}}(r). \tag{2.8.5}$$

Furthermore, by Corollary 2.4.6,

$$M_l^{\mathcal{R}} \leq C^{\mathcal{R}} M_l^A N^{\mathcal{R}}. \tag{2.8.6}$$

2.9 Representations: special correspondences

In this section, we discuss some special cases of the crossed product construction, based on Theorem 2.8.1. In the first part, we are concerned with a general algebra and group, and make the correspondence between (covariant) representations more explicit in a number of cases. In the second part, we consider Banach algebra dynamical systems where the algebra is trivial. This leads, amongst others, to what could be called group Banach algebras associated with a class of Banach spaces. The third part covers the case of a trivial group. Here the machinery as developed in the previous sections is not necessary, and Theorem 2.8.1, although applicable, does, in fact, not yield optimal results. In this case the crossed product is merely the completion of a quotient of the algebra, and the correspondence between representations is then standard, but we have nevertheless included the results for the sake of completeness of the presentation.

2.9.1 General algebra and group

Theorem 2.8.1 gives, for each class $\mathcal{X}$ of Banach spaces, a bijection between non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) and non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ in spaces from $\mathcal{X}$. By definition, a continuous covariant representation (π, U) is $\mathcal{R}$-continuous if there exists a constant C such that $\|\pi \rtimes U(f)\| \leq C \sup_{(\rho,V)\in\mathcal{R}} \|\rho \rtimes V(f)\|$, for all $f \in C_c(G, A)$. One would like to make this condition more explicit in terms of $\|\pi\|$ and $\|U_r\|$, for $r \in G$. For certain situations, this is indeed feasible (possibly by also restricting the maps $\mathcal{I}^{\mathcal{R}}$ and $\mathcal{S}^{\mathcal{R}}$ in Theorem 2.8.1 to suitable subsets of their domains) on basis of the estimates in Remark 2.8.4. Our basic theorem in this vein is the following.

Theorem 2.9.1. *Let (A, G, α) be a Banach algebra dynamical system, where, for each $\varepsilon > 0$, A has a $(1+\varepsilon)$-bounded approximate left identity. Let $\mathcal{Z}$ be a neighbourhood basis of $e \in G$ contained in a fixed compact set, let $\nu : G \to [0, \infty)$ be bounded on compact sets and satisfy $\inf_{V\in\mathcal{Z}} \sup_{r\in V} \nu(r) = 1$. Let $\mathcal{R}$ be a non-empty class of non-degenerate continuous covariant representations of (A, G, α), such that, for $(\pi, U) \in \mathcal{R}$, π is contractive and $\|U_r\| \leq \nu(r)$, for all $r \in G$.*

Let $\mathcal{X}$ be a class of Banach spaces, and suppose that $\mathcal{R}$ contains the class $\mathcal{R}'$, consisting of all non-degenerate continuous covariant representations (π, U) of (A, G, α) in spaces from $\mathcal{X}$, where π is contractive and $\|U_r\| \leq \nu(r)$, for all $r \in G$.

If $\mathcal{R}'$ is non-empty, then the map $(\pi, U) \mapsto (\pi \rtimes U)^{\mathcal{R}}$ is a bijection between $\mathcal{R}'$ and the non-degenerate contractive representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ in spaces from $\mathcal{X}$. This map preserves the set of closed invariant subspaces, as well as the Banach space of bounded intertwining operators between two elements of $\mathcal{R}'$.

If (A, G, α) and $\mathcal{R}$ are involutive, then this bijection preserves the property of being involutive.

Proof. We use Theorem 2.8.1. Suppose that $(\pi, U) \in \mathcal{R}' \subset \mathcal{R}$, then certainly (π, U) is $\mathcal{R}$-continuous, so that $\mathcal{R}' \subset \mathrm{Covrep}^{\mathcal{R}}_{\mathrm{nd,c}}((A, G, \alpha), \mathcal{X})$. Hence the results of that theorem are applicable, and we must show that $\mathcal{I}^{\mathcal{R}}$ and $\mathcal{S}^{\mathcal{R}}$ are bijections between $\mathcal{R}'$ and the non-degenerate contractive representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ in the elements of $\mathcal{X}$, which form a subset of $\mathrm{Rep}_{\mathrm{nd,b}}((A \rtimes_\alpha G)^{\mathcal{R}}, \mathcal{X})$. Suppose that $(\pi, U) \in \mathcal{R}' \subset \mathcal{R}$, then $\mathcal{I}^{\mathcal{R}}((\pi, U)) = (\pi \rtimes U)^{\mathcal{R}}$ is obviously contractive by the very definition of $\sigma^{\mathcal{R}}$ and $(A \rtimes_\alpha G)^{\mathcal{R}}$ in Definition 2.3.2. Conversely, suppose that T is a non-degenerate contractive representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ in a space from $\mathcal{X}$. In the notation of Remark 2.8.4, we have $C^{\mathcal{R}} \leq 1$, and $M_l^A \leq 1$. By definition of $\nu^{\mathcal{R}}$ we have $\nu^{\mathcal{R}} \leq \nu$, so the condition on ν implies that $N^{\mathcal{R}} \leq 1$. From (2.8.6), we then conclude that $M_l^{\mathcal{R}} \leq 1$. Thus $\mathcal{S}^{\mathcal{R}}(T) = (\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is not only known to be a non-degenerate covariant representation of (A, G, α) in a space from $\mathcal{X}$ by Theorem 2.8.1, but in addition we know from (2.8.4) that $\left\|\overline{T} \circ i_A^{\mathcal{R}}\right\| \leq 1$, and from (2.8.5) that $\left\|(\overline{T} \circ i_G^{\mathcal{R}})(r)\right\| \leq \nu^{\mathcal{R}}(r) \leq \nu(r)$, for all $r \in G$, so $\mathcal{S}^{\mathcal{R}}(T) \in \mathcal{R}'$. This settles the main part of the present theorem, and the rest is immediate from Theorem 2.8.1. □

As a particular case of Theorem 2.9.1, we let $\mathcal{R}$ and $\mathcal{R}'$ coincide, and we specialize to $\nu \equiv 1$. Note that this condition $\|U_r\| \leq 1$, for all $r \in G$, is equivalent to requiring that U is isometric. Thus we obtain the following.

Theorem 2.9.2. *Let (A, G, α) be a Banach algebra dynamical system, where, for each $\varepsilon > 0$, A has a $(1 + \varepsilon)$-bounded approximate left identity. Let $\mathcal{X}$ be a class of Banach spaces, and let $\mathcal{R}$ consist of all non-degenerate continuous covariant representations (π, U) of (A, G, α) in spaces from $\mathcal{X}$, such that π is contractive and U_r is an isometry, for all $r \in G$.*

If $\mathcal{R}$ is non-empty, then the map $(\pi, U) \mapsto (\pi \rtimes U)^{\mathcal{R}}$ is a bijection between $\mathcal{R}$ and the non-degenerate contractive representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ in spaces from $\mathcal{X}$. This map preserves the set of closed invariant subspaces, as well as the Banach space of bounded intertwining operators between two elements of $\mathcal{R}$.

If (A, G, α) and $\mathcal{R}$ are involutive, then this bijection preserves the property of being involutive.

Specializing Theorem 2.9.2 in turn to the involutive case yields the following. We recall from the third part of Remark 2.3.3 that $(A \rtimes_\alpha G)^{\mathcal{R}}$ is a C^*-algebra if (A, G, α) and $\mathcal{R}$ are involutive.

Theorem 2.9.3. *Let (A, G, α) be a Banach algebra dynamical system, where A is a Banach algebra with bounded involution and where G acts as involutive automorphisms on A. Assume that, for each $\varepsilon > 0$, A has a $(1+\varepsilon)$-bounded approximate left* *identity. Let $\mathcal{H}$ be a class of Hilbert spaces, and let $\mathcal{R}$ consist of all non-degenerate continuous covariant representations (π, U) of (A, G, α) in elements of $\mathcal{H}$, such that π is contractive and involutive, and U_r is unitary, for all $r \in G$.*

If $\mathcal{R}$ is non-empty, then the map $(\pi, U) \mapsto (\pi \rtimes U)^{\mathcal{R}}$ is a bijection between $\mathcal{R}$ and the non-degenerate involutive representations of the C^*-algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ in spaces from $\mathcal{H}$. This map preserves the set of closed invariant subspaces, as well as the Banach space of bounded intertwining operators between two elements of $\mathcal{R}$.

Remark 2.9.4. Note that Theorem 2.9.3 applies to all C^*-dynamical systems, since then A has a 1-bounded approximate left identity. In that case, if $\mathcal{R}$ is non-empty, then $(A \rtimes_\alpha G)^{\mathcal{R}}$ can be considered as the C^*-crossed product associated with the C^*-dynamical system (A, G, α) and the Hilbert spaces from $\mathcal{H}$. If $\mathcal{H}$ consists of *all* Hilbert spaces, then the associated C^*-algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ is commonly known as *the* crossed product $A \rtimes_\alpha G$, as in [51]. Surely $\mathcal{R}$ is then non-empty, since it contains the zero representation on the zero space. However, more is true: the Gelfand-Naimark theorem furnishes a faithful non-degenerate involutive representation of A in a Hilbert space, and then [51, Lemma 2.26] provides a covariant involutive representation of (A, G, α) (which is non-degenerate by [51, Lemma 2.17]), of which the integrated form is a faithful representation of $C_c(G, A)$. As a consequence, $\sigma^{\mathcal{R}}$ is then an algebra norm on $C_c(G, A)$, rather than a seminorm, and the quotient construction as in the present paper for the general case is then not necessary.

We conclude this section with a preparation for the sequel [22], where we will show that under certain conditions $(A \rtimes_\alpha G)^{\mathcal{R}}$ is (isometrically) isomorphic to the Banach algebra $L^1(G, A)$ with a twisted convolution product. With this in place, we will then also be able to show how well-known results about (bi)-modules for $L^1(G)$ ([20, Assertion VI.1.32], [24, Proposition 2.1]) fit into the general framework of crossed products of Banach algebras.

The preparatory result we will then require is the following; the function $\nu^{\mathcal{R}}$ figuring in it is defined in Remark 2.8.4.

Theorem 2.9.5. *Let (A, G, α) be a Banach algebra dynamical system, where A has a bounded approximate left identity. Let $D \geq 0$, and let $\mathcal{R}$ be a non-empty class of non-degenerate continuous covariant representations (π, U) of (A, G, α), such that $\nu^{\mathcal{R}}(r) \leq D$, for all $r \in G$. Assume that there exists $C_1 \geq 0$ such that $\|f\|_{L^1(G,A)} \leq C_1 \sup_{(\pi,U)\in\mathcal{R}} \|\pi \rtimes U(f)\|$, for all $f \in C_c(G, A)$.*

Let $\mathcal{X}$ be a class of Banach spaces. Then the map $(\pi, U) \mapsto (\pi \rtimes U)^{\mathcal{R}}$ is a bijection between the non-degenerate continuous covariant representations of (A, G, α) in spaces from $\mathcal{X}$ for which there exists a constant C_U, such that $\|U_r\| \leq C_U$, for all $r \in G$, and the non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ in spaces from $\mathcal{X}$. This map preserves the set of closed invariant subspaces, as well as the Banach space of bounded intertwining operators between two elements of $\mathcal{R}$.

If (A, G, α) and $\mathcal{R}$ are involutive, then this bijection preserves the property of being involutive.

Proof. We apply Theorem 2.8.1 and show that $\mathcal{I}^{\mathcal{R}}$ and $\mathcal{S}^{\mathcal{R}}$ map the sets of (covariant) representations as described in the present theorem into each other. Suppose

(π, U) is a non-degenerate continuous covariant representations of (A, G, α) in a space from $\mathcal{X}$ for which U is uniformly bounded. Then by (2.3.3), the assumption implies that, for all $f \in C_c(G, A)$,

$$\|\pi \rtimes U(f)\| \leq \|\pi\| \, D \, \|f\|_{L^1(G,A)} \leq \|\pi\| \, D C_1 \sigma^{\mathcal{R}}(f),$$

and so $\pi \rtimes U$ is $\mathcal{R}$-continuous and hence induces a non-degenerate bounded representation $(\pi \rtimes U)^{\mathcal{R}}$ of $(A \rtimes_\alpha G)^{\mathcal{R}}$.

Conversely, let T be a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ in a space from $\mathcal{X}$. Then $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is not only known to be a non-degenerate continuous covariant representation (π, U) of (A, G, α), by Theorem 2.8.1, but in addition (2.8.5), together with $\nu^{\mathcal{R}}(r) \leq D$ for all $r \in G$, shows that $\overline{T} \circ i_G$ is uniformly bounded. $\square$

Remark 2.9.6. Under the hypotheses of Theorem 2.9.5, (2.3.3) implies that we also have $\sigma^{\mathcal{R}}(f) \leq C^{\mathcal{R}} D \, \|f\|_{L^1(G,A)}$, so that $\sigma^{\mathcal{R}}$ and $\| \, . \, \|_{L^1(G,A)}$ are equivalent algebra norms on $C_c(G, A)$. As a consequence, $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $L^1(G, A)$ are isomorphic Banach algebras, and the non-degenerate continuous covariant representations (π, U) in spaces from $\mathcal{X}$, as described in the theorem, are in bijection with the non-degenerate bounded representations of $L^1(G, A)$ in the elements of $\mathcal{X}$. The questions when the condition $\|f\|_{L^1(G,A)} \leq C_1 \sigma^{\mathcal{R}}(f)$ is actually satisfied, and when $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $L^1(G, A)$ are even isometrically isomorphic Banach algebras, will be tackled in the sequel ([22]), to which we also postpone further discussion.

2.9.2 Trivial algebra: group Banach algebras

We now specialize the results of Theorem 2.9.1 to the case where the algebra is equal to the field $\mathbb{K}$, and the group acts trivially on it. We start by making some preliminary remarks.

The general representation of $\mathbb{K}$ in a Banach space X is given by letting $\lambda \in \mathbb{K}$ act as λP, where $P \in B(X)$ is an idempotent. Therefore, the only non-degenerate representation of $\mathbb{K}$ in X is the canonical one, $\mathrm{can}_X : \mathbb{K} \to B(X)$, obtained for $P = \mathrm{id}_X$. As a consequence, the non-degenerate covariant representations of $(\mathbb{K}, G, \mathrm{triv})$ in a given Banach space X are in bijection with the strongly continuous representations of G in that Banach space, by letting (can_X, U) correspond to U. Likewise, the non-degenerate involutive continuous covariant representations of $(\mathbb{K}, G, \mathrm{triv})$ in a given Hilbert space are in bijection with the unitary strongly continuous representations of G in that Hilbert space.

The Banach algebra dynamical system $(\mathbb{K}, G, \mathrm{triv})$ has a non-degenerate continuous covariant representation in each Banach space X, with G acting as isometries, namely, by letting the field act as scalars and letting the group act trivially. Likewise, there is a non-degenerate involutive continuous covariant representation of $(\mathbb{K}, G, \mathrm{triv})$ in each Hilbert space, with G acting as unitaries. Therefore, the hypothesis in the theorems in Section 2.9.1 that certain classes of non-degenerate continuous covariant representations are non-empty is sometimes redundant. Furthermore, the hypothesis on the existence of a suitable bounded approximate left identity in $\mathbb{K}$ is obviously always satisfied.

We introduce some shorthand notation. If $\mathcal{R}$ is a class of strongly continuous representations of G, then $\widetilde{\mathcal{R}} := \{(\mathrm{can}_{X_U}, U) : U \in \mathcal{R}\}$ is a uniformly bounded class of continuous covariant representations of $(\mathbb{K}, G, \mathrm{triv})$ precisely if there exists a function $\nu : G \to [0, \infty)$, which is bounded on compact subsets of G, and such that $\|U_r\| \leq \nu(r)$, for all $U \in \mathcal{R}$, and all $r \in G$. In that case, the associated crossed product $(\mathbb{K} \rtimes_{\mathrm{triv}} G)^{\widetilde{\mathcal{R}}}$ is defined, but we will write $(\mathbb{K} \rtimes_{\mathrm{triv}} G)^{\mathcal{R}}$ for short. Thus $(\mathbb{K} \rtimes_{\mathrm{triv}} G)^{\mathcal{R}}$ is obtained by starting with $C_c(G)$ in its usual convolution structure, introducing the seminorm

$$\sigma^{\mathcal{R}}(f) = \sup_{U \in \mathcal{R}} \left\| \int_G f(s) U_s \, ds \right\| \quad (f \in C_c(G)),$$

and completing $C_c(G)/\ker(\sigma^{\mathcal{R}})$ in the norm induced on this quotient by $\sigma^{\mathcal{R}}$. As before, we let $q^{\mathcal{R}}$ denote the canonical map from $C_c(G)$ into $(\mathbb{K} \rtimes_{\mathrm{triv}} G)^{\mathcal{R}}$. If U is a strongly continuous representation of G, then we let $U(f) = \int_G f(s) U_s \, ds$, which corresponds to $\mathrm{can}_{X_U} \rtimes U(f)$ in the previous sections. Then U will be called $\mathcal{R}$-continuous if there exists a constant C such that $\|U(f)\| \leq C \sigma^{\mathcal{R}}(f)$, for all $f \in C_c(G)$; this corresponds to $\mathrm{can}_{X_U} \rtimes U$ being $\widetilde{R}$-continuous. In that case, there is an associated bounded representation of $(\mathbb{K} \rtimes_{\mathrm{triv}} G)^{\mathcal{R}}$, denoted by $U^{\mathcal{R}}$ rather than $(\mathrm{can}_{X_U} \rtimes U)^{\mathcal{R}}$, which is given on the dense subalgebra $q^{\mathcal{R}}(C_c(G))$ of $(\mathbb{K} \rtimes_{\mathrm{triv}} G)^{\mathcal{R}}$ by

$$U^{\mathcal{R}}(q^{\mathcal{R}}(f)) = \int_G f(s) U_s \, ds \quad (f \in C_c(G)).$$

With these notations, Theorems 2.9.1 specializes to the following result.

Theorem 2.9.7. *Let G be a locally compact group. Let $\mathcal{Z}$ be a neighbourhood basis of $e \in G$ contained in a fixed compact set, let $\nu : G \to [0, \infty)$ be bounded on compact sets and satisfy $\inf_{V \in \mathcal{Z}} \sup_{r \in V} \nu(r) = 1$. Let $\mathcal{R}$ be a non-empty class of strongly continuous representations of G on Banach spaces, such that, for $U \in \mathcal{R}$, $\|U_r\| \leq \nu(r)$, for all $r \in G$.*

Let $\mathcal{X}$ be a class of Banach spaces, and suppose that $\mathcal{R}$ contains the class $\mathcal{R}'$, consisting of all strongly continuous representations U of G in spaces from $\mathcal{X}$, such that $\|U_r\| \leq \nu(r)$, for all $r \in G$.

Then the map which sends $U \in \mathcal{R}'$ to $U^{\mathcal{R}}$ is a bijection between $\mathcal{R}'$ and the non-degenerate contractive representations of $(\mathbb{K} \rtimes_{\mathrm{triv}} G)^{\mathcal{R}}$ in the Banach spaces from $\mathcal{X}$. This map preserves the set of closed invariant subspaces, as well as the Banach space of bounded intertwining operators between two elements of $\mathcal{R}'$.

If all elements from $\mathcal{R}$ are unitary strongly continuous representations, then this bijection lets unitary strongly continuous representations of G in $\mathcal{R}'$ correspond to involutive representations of the C^*-algebra $(\mathbb{C} \rtimes_{\mathrm{triv}} G)^{\mathcal{R}}$.

Specializing the above result to $\mathcal{R} = \mathcal{R}'$ and $\nu \equiv 1$ (or Theorem 2.9.2 to the case of the trivial algebra), we obtain the following.

Theorem 2.9.8. *Let G be a locally compact group. Let $\mathcal{X}$ be a non-empty class of Banach spaces, and let $\mathcal{R}$ be the class of all isometric strongly continuous representations of G in spaces from $\mathcal{X}$. Then the map which sends $U \in \mathcal{R}$ to $U^{\mathcal{R}}$ is a bijection between $\mathcal{R}$ and the non-degenerate contractive representations of the Banach algebra $(\mathbb{K} \rtimes_{\mathrm{triv}} G)^{\mathcal{R}}$ in the Banach spaces from $\mathcal{X}$. This map preserves the set of closed invariant subspaces, as well as the Banach space of bounded intertwining operators.*

The following is a consequence of Theorem 2.9.3.

Theorem 2.9.9. *Let G be a locally compact group, let $\mathcal{H}$ be a non-empty class of Hilbert spaces, and let $\mathcal{R}$ consists of all unitary strongly continuous representations of G in the Hilbert spaces from $\mathcal{H}$.*

Then the map which sends $U \in \mathcal{R}$ to $U^{\mathcal{R}}$ is a bijection between $\mathcal{R}$ and the non-degenerate involutive representations of the C^*-algebra $(\mathbb{K} \rtimes_{\mathrm{triv}} G)^{\mathcal{R}}$ in the Hilbert spaces from $\mathcal{H}$. This map preserves the set of closed invariant subspaces, as well as the Banach space of bounded intertwining operators between two elements of $\mathcal{R}$.

Remark 2.9.10. The Banach algebra in Theorem 2.9.8 could be called the group Banach algebra $\mathcal{B}_{\mathcal{X}}(G)$ of G associated with the (isometric strongly continuous representations of G in the) Banach spaces from $\mathcal{X}$. As explained in the Introduction, these algebras, and their possible future role in decomposition theory for group representations, were part of the motivation underlying the present paper. The C^*-algebra in Theorem 2.9.9 is the group Banach algebra $\mathcal{B}_{\mathcal{H}}(G)$, which in this case has additional structure as a C^*-algebra. If $\mathcal{H}$ consists of all Hilbert spaces, then the group Banach algebra $\mathcal{B}_{\mathcal{H}}(G)$ is, of course, what is commonly known as $C^*(G)$, "the" group C^*-algebra of G.

2.9.3 Trivial group: enveloping algebras

We conclude with a few remarks on the case where the group is equal to the trivial group, $\{e\}$, acting trivially on A. In this situation $C_c(\{e\}, A) \cong A$ as abstract algebras, so, if $\mathcal{R}$ is a class of representations of A in Banach spaces, for which there exists a constant C, such that $\|\pi\| \leq C$, for all $\pi \in \mathcal{R}$, then one naturally associates a uniformly bounded class of continuous covariant representations of $(A, \{e\}, \mathrm{triv})$ with $\mathcal{R}$, and, with obvious notational convention, constructs the crossed product $(A \rtimes_{\mathrm{triv}} \{e\})^{\mathcal{R}}$. This crossed product is simply the completion of $A/\ker(\sigma^{\mathcal{R}})$ in the norm corresponding to the seminorm $\sigma^{\mathcal{R}}$ on A, defined by $\sigma^{\mathcal{R}}(a) = \sup_{\pi \in \mathcal{R}} \|\pi(a)\|$, for $a \in A$.

In principle, one could apply Theorem 2.9.1 in this situation, but then one would need to require A to have a bounded left approximate identity. The reason underlying this is that, in general, there are no homomorphisms of the algebra or the group into the crossed product, so that the most natural idea to obtain representations of algebra and group from a representation of the crossed product, namely, to compose a given representation of the crossed product with such homomorphisms, will not work in general. In our approach in previous sections, the left centralizer algebra of the crossed product, into which the algebra and group do map, provided an alternative, but then one needs a bounded left approximate identity of the algebra, in order to be able to construct a representation of the left centralizer algebra from a (non-degenerate) representation of the crossed product. In the present case of a trivial group, however, one needs only a homomorphism of the algebra A into the crossed product, and since this is a completion of $A/\ker(\sigma^{\mathcal{R}})$, this clearly exists and the machinery we had to employ in previous sections is now not required. Also, the non-degeneracy of representations (needed to construct representations of the left centralizer algebra) is no longer an issue. One simply applies Lemma 2.2.20, and thus obtains the following elementary and well-known theorem for the crossed product $(A \rtimes_{\mathrm{triv}} \{e\})^{\mathcal{R}}$, which we include for the sake of completeness.

Theorem 2.9.11. *Let A be a Banach algebra, and let $\mathcal{R}$ be a non-empty uniformly bounded class of representations of A in Banach spaces. Let $\sigma^{\mathcal{R}}(a) = \sup_{\pi\in\mathcal{R}} \|\pi(a)\|$, for $a \in A$, denoted the associated seminorm, and let $A^{\mathcal{R}}$ be the completion of $A/\ker(\sigma^{\mathcal{R}})$ in the norm corresponding to $\sigma^{\mathcal{R}}$ on A.*

Let $\mathcal{X}$ be a non-empty class of Banach spaces, and say that a bounded representation π of A in a space from $\mathcal{X}$ is $\mathcal{R}$-continuous if there exists a constant C, such that $\|\pi(a)\| \leq C\sigma^{\mathcal{R}}(a)$, for all $a \in A$. In that case, define the norm of π as the minimal such C.

Then the $\mathcal{R}$-continuous representations of A in the spaces from $\mathcal{X}$ correspond naturally with the bounded representations of $A^{\mathcal{R}}$ in spaces from $\mathcal{X}$. This correspondence preserves the norms of the representations, the set of closed invariant subspaces, and the Banach spaces of bounded intertwining operator.

If A has a (possibly unbounded) involution, and if all elements of $\mathcal{R}$ are involutive bounded representations, then this natural correspondence sets up a bijection between the $\mathcal{R}$-continuous involutive bounded representations of A in the Hilbert spaces in $\mathcal{X}$, and the involutive representations of the C^*-algebra $A^{\mathcal{R}}$ in those spaces.

If A is an involutive Banach algebra with an isometric involution and a bounded approximate identity, and $\mathcal{R}$ consists of all involutive representations in Hilbert spaces, which is uniformly bounded since all such representations are contractive by [10, Proposition 1.3.7], then the crossed product $A^{\mathcal{R}}$ is generally known as the enveloping C^*-algebra of A as described in [10, Section 2.7].

Acknowledgements

The authors thank Miek Messerschmidt for helpful comments and suggestions.

Chapter 3

Positive representations of finite groups in Riesz spaces

This chapter is to appear in International Journal of Mathematics as: M. de Jeu and M. Wortel, "Positive representations of finite groups in Riesz spaces". It is available as arXiv:1109.4505.

Abstract. In this paper, which is part of a study of positive representations of locally compact groups in Banach lattices, we initiate the theory of positive representations of finite groups in Riesz spaces. If such a representation has only the zero subspace and possibly the space itself as invariant principal bands, then the space is Archimedean and finite dimensional. Various notions of irreducibility of a positive representation are introduced and, for a finite group acting positively in a space with sufficiently many projections, these are shown to be equal. We describe the finite dimensional positive Archimedean representations of a finite group and establish that, up to order equivalence, these are order direct sums, with unique multiplicities, of the order indecomposable positive representations naturally associated with transitive G-spaces. Character theory is shown to break down for positive representations. Induction and systems of imprimitivity are introduced in an ordered context, where the multiplicity formulation of Frobenius reciprocity turns out not to hold.

3.1 Introduction and overview

The theory of unitary group representations is well developed. Apart from its intrinsic appeal, it has been stimulated in its early days by the wish, originating from quantum theory, to understand the natural representations of symmetry groups of physical systems in L^2-spaces. Such symmetry groups do not only yield natural unitary representations, but they have natural representations in other Banach spaces as well. For example, the orthogonal group in three dimensions does not only act on the L^2-functions on the sphere or on three dimensional space. It also has a natural isometric action on L^p-functions, for all p, and this action is strongly continuous for finite p. Moreover, this action is obviously positive. Thus, for finite p, these L^p-spaces, which are Banach lattices, afford a strongly continuous isometric positive representation of the orthogonal group. It is rather easy to find more examples of positive representations: whenever a group acts on a point set, then, more often than not, there is a natural positive action on various naturally associated Banach lattices of functions.

However, in spite of the plenitude of examples of strongly continuous (isometric) positive representations of groups in Banach lattices, the theory of such representations cannot compare to its unitary counterpart. Very little seems to be known. Is there, for example, a decomposition theory into indecomposables for such representations, as a positive counterpart to that for unitary representations described in, e.g., [10]? When asking such an—admittedly ambitious—question, it is important to keep in mind that the unitary theory works particularly well in separable Hilbert spaces, i.e., for representations which can all be realized in just one space: ℓ^2. Since there is a great diversity of Banach lattices, it is not clear at the time of writing whether one can expect a general answer for all these lattices with a degree of sophistication and uniformity comparable to that for the unitary representations in this single Hilbert space ℓ^2. It may be more feasible, at least for the moment, to aim at a better understanding of positive representations on specific classes of Banach lattices. For example, the results in [23] show that, in the context of a Polish group acting on a Polish space with an invariant measure, it is indeed possible to decompose—in terms of Banach bundles rather than in terms of direct integrals as in the unitary case—the corresponding isometric positive representation in L^p-spaces ($1 \leq p < \infty$) into indecomposable isometric positive representations. To our knowledge, this is the only available decomposition result at this time. Since this result covers only representations originating from an action on the underlying point set, we still cannot decide whether a general (isometric) positive representation of a (Polish) group in such Banach lattices can be decomposed into indecomposable positive representations, and more research is necessary to decide this. This, however, is already a relatively advanced issue: as will become clear below, it is easy to ask very natural basic questions about positive representations of locally compact groups in Banach lattices which need answering. This paper, then, is a contribution to the theory of such representations, with a hoped-for decomposition theory into indecomposable positive representations in mind as a leading and focusing theme, and starting with the obviously easiest case: the finite groups. We will now globally discuss it contents.

If a group G acts as positive operators on a Banach lattice E, then the natural question is to ask whether it is possible to decompose $E = L \oplus M$ as a G-invariant order direct sum. In that case, L and M are automatically projection bands and each other's disjoint complement. Since bands in a Banach lattice are closed, the decomposition is then automatically also topological, and the original representations splits as an order direct sum of the positive subrepresentations on the Banach lattices L and M. If such a decomposition is only trivially possible, then we will call the representation (order) indecomposable, a terminology already tacitly used above. Is it then true that every indecomposable positive representation of a finite group G in a Banach lattice is finite dimensional, as it is for unitary representations? This is not the case: the trivial group acting on $C([0,1])$, which has only trivial projection bands, provides a counterexample. Is it then perhaps true when we narrow down the class of spaces to better behaved ones, and ask the same question for Banach lattices with the projection property? After all, since bands are now complemented by their disjoint complement, they seem close to Hilbert spaces where the proof for the corresponding statement in the unitary case is a triviality, and based on this complementation property. Indeed, if $x \neq 0$ is an element of the Hilbert space under consideration where the finite group G acts unitarily and irreducibly, then the orbit $G \cdot x$ spans a finite dimensional, hence closed, nonzero subspace which is clearly invariant and invariantly complemented. Hence the orbit spans the space, which must be finite dimensional. In an ordered context this proof breaks down. Surely, one can consider the band generated by the orbit of a nonzero element, which is invariantly complemented in an order direct sum if the space is assumed to have the projection property. Hence this band is equal to the space, but since there is no guarantee that it is finite dimensional, once cannot reach the desired conclusion along these lines. We have not been able to find an answer to this finite dimensionality question for Banach lattices with the projection property in the literature, nor could a number of experts in positivity we consulted provide an answer. The best available result in this vein seems to be [43, Theorem III.10.4], which implies as a special case that a positive representation of a finite group in a Banach lattice with only trivial invariant closed ideals is finite dimensional. Still, this does not answer our question concerning the finite dimensionality of indecomposable positive representations of a finite group in a Banach lattice with the projection property. The reason is simple: unless one assumes that the lattice has order continuous norm, one cannot conclude that there are only trivial invariant closed ideals from the fact that there are only trivial invariant bands. On the other hand: there are no obvious infinite dimensional counterexamples in sight, and one might start to suspect that there are none. This is in fact the case, and even more holds true: a positive representation of a finite group in a Riesz space, with the property that the only invariant principal bands are $\{0\}$ and possibly the space itself, is in a finite dimensional Archimedean space, cf. Theorem 3.3.14 below. As will have become obvious from the previous discussion, such a result is no longer a triviality in an ordered context. It provides an affirmative answer to our original question because, for Banach lattices with the projection property, an invariant principal band is an invariant projection band. It also implies the aforementioned result that a positive representation of a finite group in a Banach lattice with only trivial invariant closed ideals is finite dimensional. Indeed, an invariant principal band is then an invariant closed ideal.

Thus, even though our original question was in terms of Banach lattices, and motivated by analytical unitary analogies, an answer can be provided in a more general, topology free context. For finite groups, this is—in fact—perhaps not too big a surprise. Furthermore, we note that the hypothesis in this finite dimensionality theorem is not the triviality of all invariant order decompositions, but rather the absence of a nontrivial G-invariant object, without any reference to this being invariantly complemented in an order direct sum or not. It thus becomes clear that it is worthwhile to not only consider the existence of nontrivial invariant projection bands (which is the same as the representation being (order) decomposable), but to also consider the existence of nontrivial invariant ideals, nontrivial invariant bands, etc., for positive representations in arbitrary Riesz spaces, and investigate the interrelations between the corresponding notions of irreducibility. In the unitary case, indecomposability and irreducibility (for which there is only one reasonable notion) coincide, but in the present ordered context this need not be so. Nevertheless, for finite groups acting positively in spaces with sufficiently many projections, the most natural of these notions of irreducibility are all identical and coincide with (order) indecomposability, cf. Theorem 3.3.16. Again, whereas the corresponding proof for the unitary case is a triviality, this is not quite so obvious in an ordered context.

As will become apparent in this paper, it is possible to describe all finite dimensional positive Archimedean representations of a finite group, indecomposable or not. Once this is done, it is not too difficult to show that such finite dimensional positive representations can be decomposed uniquely into indecomposable positive representations, and that, up to order equivalence, all such indecomposable positive representations arise from actions of the group on transitive G-spaces, cf. Corollary 3.4.11 and Theorem 3.4.10. Since this decomposition into irreducible positive representations with multiplicities is so reminiscent of classical linear representation theory theory for finite groups, and to Peter-Weyl theory for compact groups, one might wonder whether parts of character theory also survive. This is hardly the case. For finite groups with only normal subgroups, such as finite abelian groups, there is still a bijection between characters and order equivalence classes of finite dimensional indecomposable positive Archimedean representations, cf. Corollary 3.4.14, but we provide counterexamples to a number of other results as they would be natural to conjecture.

Finally, we consider induction and systems of imprimitivity in an ordered context. As long as topology is not an issue, this can be done from a categorical point of view for arbitrary groups and arbitrary subgroups. Even though the constructions are fairly routine, we have included the material, not only as a preparation for future more analytical considerations, but also because there are still some differences with the linear theory. For example, Frobenius reciprocity no longer holds in its multiplicity formulation.

After this global overview we emphasize that, even though this paper contains a basic finite dimensionality result and provides reasonably complete results for finite dimensional positive Archimedean representations of finite groups (in analytical terms: for positive representations of finite groups in $C(K)$ for K finite), the basic decomposition issue for positive representations of a finite group in infinite dimensional Banach lattices is still open. At the time of writing, the only results in this direction seem to be the specialization to finite groups of the results in [23] for L^p-spaces, and of those in [21], which is concerned with Jordan-Hölder theory for finite chains of various invariant order structures in Riesz spaces. As long as the group is only finite, a more comprehensive answer seems desirable.

The structure of this paper is as follows.

In Section 3.2 we introduce the necessary notation and definitions, and we recall a folklore result on transitive G-spaces. Then, in Section 3.3, we investigate the relations between various notions of irreducibility as already mentioned above. We then establish one of the main results of this paper, Theorem 3.3.14, stating, amongst others, that a principal band irreducible positive representation of a finite group is always finite dimensional. The proof is by induction on the order of the group, and uses a reasonable amount of general basic theory of Riesz spaces. We then continue by examining the structure of finite dimensional positive Archimedean representations of a finite group in Section 3.4. Any such space is lattice isomorphic to $\mathbb{R}^n$, for some n, and its group $\mathrm{Aut}^+(\mathbb{R}^n)$ of lattice automorphisms is a semidirect product of S_n and the group of multiplication (diagonal) operators, a result which also follows from [32, Theorem 3.2.10], but which we prefer to derive by elementary means in our context. Armed with this information we can completely determine the structure of a positive representations of a finite group in $\mathbb{R}^n$ in Theorem 3.4.5: such a positive representation is given by a representation into $S_n \subset \mathrm{Aut}^+(\mathbb{R}^n)$, called a permutation representation, and a single multiplication operator. We then determine when two positive representations in $\mathbb{R}^n$ are order equivalent, which turns out to be the case precisely when their permutation parts are conjugate. Consequently, in the end, the finite dimensional positive Archimedean representations of a finite group can be described in terms of actions of the group on finite sets. The decomposition result and the description of indecomposable positive representations already mentioned above then follow easily. The rest of Section 3.4 is concerned with showing that character theory does not survive in an ordered context. Finally, in Section 3.5, we develop the theory of induction and systems of imprimitivity in the ordered setting, and show that Frobenius reciprocity does not hold in its multiplicity formulation.

3.2 Preliminaries

In this section we will discuss some preliminaries about automorphisms of Riesz spaces, representations, order direct sums of representations and G-spaces. All Riesz spaces in this paper are real. For positive representations in spaces admitting a complexification it is easy, and left to the reader, to formulate the corresponding complex result and derive it from the real case.

Let E be a not necessarily Archimedean Riesz space. If $D \subset E$ is any subset, then the band generated by D is denoted by $\{D\}$, and the disjoint complement of D is denoted by D^d. If T is a lattice automorphism of E, then $\{TD\} = T\{D\}$ and $T(D^d) = (TD)^d$. The group of lattice automorphisms of E is denoted by $\mathrm{Aut}^+(E)$.

In this paper $\mathbb{R}^n$ is always equipped with the coordinatewise ordering.

Definition 3.2.1. Let G be a group and E a Riesz space. A *positive representation* of G in E is a group homomorphism $\rho\colon G \to \mathrm{Aut}^+(E)$.

For typographical reasons, we will write ρ_s instead of $\rho(s)$, for $s \in G$.

If $(E_i)_{i\in I}$ is a collection of Riesz spaces, then the *order direct sum* of this collection, denoted $\oplus_{i\in I}E_i$, is the Riesz space with elements $(x_i)_{i\in I}$, where $x_i \in E_i$ for all $i \in I$, at most finitely many x_i are nonzero, and where $(x_i)_{i\in I}$ is positive if and only if x_i is positive for all $i \in I$. If additionally $\rho^i\colon G \to \mathrm{Aut}^+(E_i)$ is a positive representation for all $i \in I$, then the positive representation

$$\bigoplus_{i\in I}\rho^i\colon \to \mathrm{Aut}^+\left(\bigoplus_{i\in I}E_i\right),$$

the *order direct sum* of the ρ^i, is defined by $(\oplus_{i\in I}\rho^i)_s := \oplus_{i\in I}\rho^i_s$, for $s \in G$.

Let $\rho\colon G \to \mathrm{Aut}^+(E)$ be a positive representation, and suppose that $E = L \oplus M$ as an order direct sum. Then L and M are automatically projection bands with $L^d = M$ by [28, Theorem 24.3]. If both L and M are ρ-invariant, then ρ can be viewed as the order direct sum of ρ acting positively on L and M.

If $\rho\colon G \to \mathrm{Aut}^+(E)$ and $\theta\colon G \to \mathrm{Aut}^+(F)$ are positive representations on Riesz spaces E and F, respectively, then a positive map $T\colon E \to F$ is called a *positive intertwiner* between ρ and θ if $T\rho_s = \theta_s T$ for all $s \in G$, and ρ and θ are called *order equivalent* if there exists a positive intertwiner between ρ and θ which is a lattice isomorphism.

Turning to the terminology for G-spaces, we let G be a not necessarily finite group. A G-space X is a nonempty set X equipped with an action of G; it is called transitive if there is only one orbit. For $x \in X$, let G_x denote the subgroup $\{s \in G : sx = x\}$, the stabilizer of x. If X and Y are G-spaces, then X and Y are called isomorphic G-spaces if there is a bijection $\phi\colon X \to Y$, such that $s\phi(x) = \phi(sx)$ for all $s \in G$ and $x \in X$. We let $[X]$ denote the class of all G-spaces isomorphic to X.

If X is a transitive G-space and $x \in X$, then $sx \mapsto sG_x$ is a G-space isomorphism between X and G/G_x with its natural G-action. The next folklore lemma elaborates on this correspondence.

Lemma 3.2.2. *Let G be a not necessarily finite group. For each isomorphism class $[X]$ of transitive G-spaces, choose a representative X and $x \in X$. Then the conjugacy class $[G_x]$ of G_x does not depend on the choices made, and the map $[X] \mapsto [G_x]$ is a bijection between the isomorphism classes of transitive G-spaces and the conjugacy classes of subgroups of G.*

Proof. It is easy to see that the map is well-defined and surjective. For injectivity, let X and Y be transitive G-spaces, such that $[G_x] = [G_y]$ for some $x \in X$ and $y \in Y$. We have to show that $[X] = [Y]$, or equivalently, $G/G_x \cong G/G_y$. By assumption $G_x = rG_yr^{-1}$ for some $r \in G$, and the map $sG_x \mapsto sG_xr = srG_y$ is then an isomorphism of G-spaces between G/G_x and G/G_y. $\square$

3.3 Irreducible and indecomposable representations

In the theory of unitary representations of groups, the nonexistence of nontrivial closed invariant subspaces is the only reasonable notion of irreducibility of a representation, and it coincides with the natural notion of indecomposability of a representation. In a purely linear context, irreducibility and indecomposability of group representations need not coincide, however, and the same is true in an ordered context where, in addition, there are several natural notions of irreducibility. In this section, we are concerned with the relations between the various notions and we establish a basic finite dimensionality result, Theorem 3.3.14. This is then used to show that, in fact, the various notions of irreducibility are equivalent for finite groups if the space has sufficiently many projections, cf. Theorem 3.3.16. We let G be a group, to begin with not necessarily finite.

Definition 3.3.1. A positive representation $\rho\colon G \to \mathrm{Aut}^+(E)$ is called *band irreducible* if a ρ-invariant band equals $\{0\}$ or E. Projection band irreducibility, ideal irreducibility, and principal band irreducibility are defined similarly, as are closed ideal irreducibility, etc., in the case of normed Riesz spaces.

Starting our discussion of the implications between the various notions of irreducibility, we note that, obviously, band irreducibility implies projection band irreducibility. If E has the projection property, then the converse holds trivially as well, since all bands are projection bands by definition, but the next example shows that this converse fails in general.

Example 3.3.2. Consider the representation of the trivial group on $C[0,1]$. Now every band is invariant, so this representation is not band irreducible, but $C[0,1]$ does not have any nontrivial projection bands, and therefore it is projection band irreducible.

For positive representations on Banach lattices, closed ideal irreducibility implies band irreducibility. If the Banach lattice has order-continuous norm, then the converse holds as well, since then all closed ideals are bands ([32, Corollary 2.4.4]), but once again this converse fails in general, as the next example shows.

Example 3.3.3. Consider $\ell^\infty(\mathbb{Z})$, the space of doubly infinite bounded sequences, and define $\rho\colon \mathbb{Z} \to \mathrm{Aut}^+(\ell^\infty(\mathbb{Z}))$ by $\rho_k(x_n) := (x_{n-k})$, the left regular representation. This representation is not closed ideal irreducible, since the space $c_0(\mathbb{Z})$ of sequences tending to zero is an invariant closed ideal. On the other hand, it is easy to see that any nonzero invariant ideal must contain the order dense subspace of finitely supported sequences, therefore ρ is band irreducible.

Finally, for positive representations on Banach lattices, ideal irreducibility obviously implies closed ideal irreducibility, but again there is an example showing that the converse fails in general.

Example 3.3.4. Consider the left regular representation of $\mathbb{Z}$ on $c_0(\mathbb{Z})$, as in the above example. Then $\ell^1(\mathbb{Z})$ is an invariant ideal, so this positive representation is not ideal irreducible, but it is closed ideal irreducible since every nonzero invariant ideal must contain the norm dense subspace of finitely supported sequences.

We continue by defining the natural notion of indecomposability for positive representations, which is order indecomposability. As usual, the order direct sum $E = L \oplus M$ is called nontrivial if $L \neq 0$ and $L \neq E$.

Definition 3.3.5. A positive representation $\rho\colon G \to \mathrm{Aut}^+(E)$ is called *order indecomposable* if there are no nontrivial ρ-invariant order direct sums $E = L \oplus M$.

We will now investigate the conditional equivalence between order indecomposability and the various notions of irreducibility.

Lemma 3.3.6. *A positive representation $\rho\colon G \to \mathrm{Aut}^+(E)$ is order indecomposable if and only if it is projection band irreducible.*

Proof. Suppose ρ is order indecomposable, and let B be a ρ-invariant projection band. We claim that B^d is ρ-invariant. For this, let $x \in (B^d)^+$ and $s \in G$. Then $(\rho_s x) \wedge y = \rho_s(x \wedge \rho_s^{-1} y) = \rho_s 0 = 0$ for all $y \in B^+$, so $\rho_s x \perp B$, i.e., $\rho_s x \in B^d$. Hence $E = B \oplus B^d$ is a ρ-invariant order direct sum, so either $B = 0$ or $B = E$.

Conversely, suppose that ρ is projection band irreducible. Let $E = L \oplus M$ be a ρ-invariant order direct sum. Then, as mentioned in the preliminaries, L and M are projection bands, and therefore $L = 0$ or $M = 0$. $\square$

Thus order indecomposability is equivalent with projection band irreducibility. We have already seen that the latter property is, in general, not equivalent with band irreducibility, but that equivalence between these two does hold (trivially) if the Riesz space has the projection property. However, if the group is finite, we will see in Lemma 3.3.9 and Theorem 3.3.16 below that these three notion are equivalent under a much milder assumption on the space, as in the following definition.

Definition 3.3.7. A Riesz space E is said to have *sufficiently many projections* if every nonzero band contains a nonzero projection band.

This notion is intermediate between the principal projection property and the Archimedean property, cf. [28, Theorem 30.4]. In order to show that it is (in particular) actually weaker than the projection property, which is the relevant feature for our discussion, we present an example of a Banach lattice which has sufficiently many projections, but not the projection property.

Example 3.3.8. Let $\Delta \subset [0, 1]$ be the Cantor set, and let $E = C(\Delta)$. Then [32, Corollary 2.1.10] shows that bands correspond to (all functions vanishing on the complement of) regularly open sets, i.e., to open sets which equal the interior of their closure, and projection bands correspond to clopen sets. The Cantor set has a basis of clopen sets, so that, in particular, every nonempty regularly open set contains a nonempty clopen set. Therefore $C(\Delta)$ has sufficiently many projections. It does not have the projection property, since $[0, 1/4) \cap \Delta \subset \Delta$ is regularly open but not closed ([48, 29.7]).

Lemma 3.3.9. *Let G be a finite group, E a Riesz space with sufficiently many projections, and $\rho\colon G \to \mathrm{Aut}^+(E)$ a positive representation. Then the following are equivalent:*

(*i*) *ρ is order indecomposable;*

(*ii*) *ρ is projection band irreducible;*

(*iii*) *ρ is band irreducible.*

Proof. (*iii*) $\Rightarrow$ (*ii*) is immediate. For (*ii*) $\Rightarrow$ (*iii*), suppose ρ is projection band irreducible. Let B_0 be a nonzero ρ-invariant band. Let $0 \neq B \subset B_0$ be a projection band. Then $\sum_{s\in G} \rho_s B$ is a projection band by [28, Theorem 30.1(ii)], and clearly it is ρ-invariant, nonzero, and contained in B_0. Therefore it must equal E, and so $B_0 = E$.

(*i*) $\Leftrightarrow$ (*ii*) follows from Lemma 3.3.6. □

This lemma will be improved significantly later on, see Theorem 3.3.16.

We will now investigate the question whether a positive representation of a finite group, satisfying a suitable notion of irreducibility, is necessarily finite dimensional. As explained in the introduction, this is not quite as obvious as it is for Banach space representations. It follows as a rather special case from [43, Theorem III.10.4] that a closed ideal irreducible positive representation of a finite group in a Banach lattice is finite dimensional, but this seems to be the only known available result in this vein. We will show, see Theorem 3.3.14, that a positive principal band irreducible representation of a finite group in a Riesz space is finite dimensional. This implies the aforementioned finite dimensionality result for Banach lattices. As a preparation, we need four lemmas.

Lemma 3.3.10. *Let G be a finite group, E a Riesz space and $\rho\colon G \to \mathrm{Aut}^+(E)$ a positive principal band irreducible representation. Then E is Archimedean.*

Proof. Suppose E is not Archimedean. Then $E \neq 0$ and there exist $x, y \in E$ such that $0 < \lambda x \leq y$ for all $\lambda > 0$. The band B generated by $\sum_{s\in G} \rho_s x$ is principal, ρ-invariant and nonzero, and therefore equals E. Let $u \geq 0$ be an element of the ideal I generated by $\sum_{s\in G} \rho_s x$. Then for some $\lambda \geq 0$,

$$u \leq \lambda \sum_{s\in G} \rho_s x = \sum_{s\in G} \rho_s(\lambda x) \leq \sum_{s\in G} \rho_s y,$$

and so $\sum_{s\in G} \rho_s y$ is an upper bound for I^+, and hence for $B^+ = E^+$, which is absurd since $E \neq 0$. Therefore E is Archimedean. □

Lemma 3.3.11. *Let E be a Riesz space with* $\dim(E) \geq 2$. *Then E contains a nontrivial principal band.*

Proof. Suppose E does not contain a nontrivial principal band. Then the trivial group acts principal band irreducibly on E, so by Lemma 3.3.10, E is Archimedean.

Furthermore E is totally ordered, otherwise there exists an element x which is neither positive nor negative and so $x^+ \notin B_{x^-} = E$. However, by [1, Exercise 1.14 (proven on page 272)], a totally ordered and Archimedean space has dimension 0 or 1, which is a contradiction. Hence E contains a nontrivial principal band. □

Lemma 3.3.12. *Let E be an Archimedean Riesz space and let $I \subset E$ be a finite dimensional ideal. Then I is a principal projection band.*

Proof. By [28, Theorem 26.11] $I \cong \mathbb{R}^n$. Let $e_1, \ldots, e_n$ be atoms that generate I. It follows that $e_1, \ldots, e_n$ are also atoms in E, and that $I = \sum_k I_{e_k}$, where I_{e_k} denotes the ideal generated by e_k. By [28, Theorem 26.4] the I_{e_k} are actually projection bands in E, and so I, as a sum of principal projection bands, is a principal projection band by [28, Chapter 4.31, page 181]. □

Lemma 3.3.13. *Let G be a finite group, E an Archimedean Riesz space, $B' \subset E$ a nonzero principal band and $\rho : G \to \mathrm{Aut}^+(E)$ a positive representation. Then there exists a nonzero principal band $B \subset B'$ such that for all $t \in G$, either $B \cap \rho_t B = 0$ or $B = \rho_t B$.*

Proof. The set $\mathcal{S} := \{S \subset G : e \in S, \bigcap_{s\in S} \rho_s B' \neq 0\}$ is partially ordered by inclusion and nonempty, since $\{e\} \in \mathcal{S}$. Pick a maximal element $M \in \mathcal{S}$, and let $B := \bigcap_{s\in M} \rho_s B'$. Then B is a principal band by [28, Theorem 48.1]. Let $t \in G$ and suppose that $B \cap \rho_t B \neq 0$. Then

$$0 \neq B \cap \rho_t B = \bigcap_{s\in M} \rho_s B' \cap \rho_t \bigcap_{s\in M} \rho_s B' = \bigcap_{r\in M\cup tM} \rho_r B',$$

and by the maximality of M we obtain $M \cup tM = M$, and so $tM \subset M$. Combined with $|tM| = |M|$ we conclude that $tM = M$, and then

$$\rho_t B = \rho_t \bigcap_{s\in M} \rho_s B' = \bigcap_{r\in tM} \rho_r B' = \bigcap_{r\in M} \rho_r B' = B.$$

□

Using these lemmas, we can now establish our main theorem on finite dimensionality.

Theorem 3.3.14. *Let G be a finite group, let E be a nonzero Riesz space and let $\rho\colon G \to \mathrm{Aut}^+(E)$ a positive principal band irreducible representation. Then E is Archimedean, finite dimensional, and the dimension of E divides the order of G.*

Proof. Lemma 3.3.10 shows that E is Archimedean. The proof is by induction on the order of G. If G is the trivial group, then E is one dimensional by Lemma 3.3.11, and we are done. Suppose, then, that the theorem holds for all groups of order strictly smaller than the order or G. If E has only trivial principal bands, then by Lemma 3.3.11 E has dimension one, and we are done again. Hence we may assume that there exists a principal band $0 \neq B' \neq E$. By Lemma 3.3.13 there exists a nonzero principal band $B \subset B' \neq E$ such that $H := \{t \in G : B = \rho_t B\}$ satisfies $H^c = \{t \in G : B \cap \rho_t B = 0\}$. It is easy to see that H is a subgroup of G, and has strictly smaller order than G: otherwise B is a nontrivial ρ-invariant principal band, contradicting the principal band irreducibility of ρ.

We will now show that ρ restricted to H is principal band irreducible on the Riesz space B. Suppose $0 \neq A \subset B$ is an H-invariant principal band of B. By [28, Theorem 48.1] $\{\sum_{s\in G} \rho_s A\}$ is a principal band, and so it is a nonzero ρ-invariant principal band of E. Hence it equals E, so using [28, Theorem 20.2(ii)] in the second step and [53, Exercise 7.7(iii)] in the third step,

$$\begin{aligned}
B &= B \cap \left\{\sum_{s\in G} \rho_s A\right\} \\
&= \left\{B \cap \sum_{s\in G} \rho_s A\right\} \\
&= \left\{\sum_{s\in G} (B \cap \rho_s A)\right\} \\
&= \left\{\sum_{s\in H} (B \cap \rho_s A) + \sum_{s\in H^c} (B \cap \rho_s A)\right\} \\
&\subset \left\{\sum_{s\in H} (B \cap \rho_s A) + \sum_{s\in H^c} (B \cap \rho_s B)\right\} \\
&\subset \left\{\sum_{s\in H} (B \cap A) + \sum_{s\in H^c} 0\right\} \\
&= \left\{\sum_{s\in H} A\right\} \\
&= A.
\end{aligned}$$

We conclude that $\rho|_H : H \to \mathrm{Aut}^+(B)$ is principal band irreducible, so B has finite dimension by the induction hypothesis. By Lemma 3.3.12, $\sum_{s\in G} \rho_s B$ is a principal band, which is nonzero and invariant, hence equal to E, and so E has finite dimension as well.

Consider the sum $\sum_{sH\in G/H} \rho_s B$. This is well defined, since $\rho_t B = B$ for $t \in H$. Moreover, if $sH \neq rH$, then $r^{-1}s \notin H$ and so $\rho_{r^{-1}s} B \cap B = 0$, implying $\rho_s B \cap \rho_r B = 0$. Therefore $\sum_{sH\in G/H} \rho_s B$ is a sum of ideals with pairwise zero intersection, which is easily seen to be a direct sum using [28, Theorem 17.6(ii)]. It follows that

$$E = \sum_{s\in G} \rho_s B = \sum_{sH\in G/H} \rho_s B = \bigoplus_{sH\in G/H} \rho_s B.$$

Therefore

$$\frac{|G|}{\dim(E)} = \frac{|G|}{|G:H|\dim(B)} = \frac{|H|}{\dim(B)} \in \mathbb{N},$$

where the last step is by the induction hypothesis. Hence the dimension of E divides the order of G as well. □

From Theorem 3.4.10, where we will explicitly describe all representations as in Theorem 3.3.14, it will also become clear that the dimension of the space divides the order of the group.

Remark 3.3.15. Note that Theorem 3.3.14 trivially implies a similar theorem for positive representations which are ideal irreducible, or which are band irreducible. It also answers our original question as mentioned in the Introduction: a positive projection band irreducible representation of a finite group in a Banach lattice with the projection property is finite dimensional. Indeed, an invariant principal band is then an invariant projection band.

When combining Theorem 3.3.14 with Lemma 3.3.9, we obtain the following result. Amongst others it shows that, under a mild condition on the space, various notions of irreducibility for a positive representation of a finite group are, in fact, the same for finite groups. It should be compared with the equality of irreducibility and indecomposability for unitary representations of arbitrary groups, and for finite dimensional representations of finite groups whenever Maschke's theorem applies. As already mentioned earlier, if a Riesz space has sufficiently many projections, it is automatically Archimedean, cf. [28, Theorem 30.4].

Theorem 3.3.16. *Let G be a finite group, E a Riesz space with sufficiently many projections and $\rho\colon G \to \mathrm{Aut}^+(E)$ a positive representation. Then the following are equivalent:*

(*i*) *ρ is order indecomposable;*

(*ii*) *ρ is projection band irreducible;*

(*iii*) *ρ is band irreducible;*

(*iv*) *ρ is ideal irreducible;*

(*v*) *ρ is principal band irreducible.*

If these equivalent conditions hold, then E is finite dimensional, and the dimension of E divides the order of G if E is nonzero.

Proof. By Lemma 3.3.9 the first three conditions are equivalent. Each of the last three conditions implies that ρ is principal band irreducible, so by Theorem 3.3.14 each of these three conditions implies that E is finite dimensional, hence lattice isomorphic to $\mathbb{R}^n$ for some n ([28, Theorem 26.11]). But then the collections of bands, ideals and principal bands in E are all the same, and hence the last three conditions are equivalent as well. The remaining statements follow from Theorem 3.3.14. □

3.4 Structure of finite dimensional positive Archimedean representations

Now that we know from Section 3.3 that positive representations of finite groups, irreducible as in Theorem 3.3.14 or 3.3.16, are necessarily in Archimedean and finite dimensional spaces, our goal is to describe the general positive finite dimensional Archimedean representations of a finite group. In such spaces, the collections of (principal) ideals, (principal) bands and projection bands are all the same, and we will use the term "irreducible positive representation" throughout this section to denote the corresponding common notion of irreducibility, which is the same as order indecomposability. We will see in Theorem 3.4.9 that positive finite dimensional Archimedean representations of a finite group split uniquely into irreducible positive representations. Furthermore, the order equivalence classes of finite dimensional irreducible positive representations are in natural bijection with the isomorphism classes of transitive G-spaces, cf. Theorem 3.4.10. The latter result can be thought of as the description of the finite dimensional Archimedean part of the order dual of a finite group. The fact that such irreducible positive representations can be realized in this way also follows from [43, Theorem III.10.4], where it is shown that strongly continuous closed ideal irreducible positive representations of a locally compact group in a Banach lattice, with compact image in the strong operator topology, can be realized on function lattices on homogeneous spaces. This general result, however, requires considerable machinery. Therefore we prefer the method below, where all follows rather easily once an explicit description of the general, not necessarily irreducible, positive representation of a finite group in a finite dimensional Archimedean space has been obtained, a result which has some relevance of its own.

Since the decomposition result below is such a close parallel to classical semisimple representation theory of finite groups, it is natural to ask whether any other features of this purely linear context survive, such as character theory. At the end of this section we show that this is, for general groups, not the case, and in the next section we will see that this is only partly so for induction.

We now proceed towards the first main step, the description of a positive finite dimensional Archimedean representation of a finite group. Since an Archimedean finite dimensional Riesz spaces is isomorphic to $\mathbb{R}^n$ for some n ([28, Theorem 26.11]), we start by describing its group $\mathrm{Aut}^+(\mathbb{R}^n)$ of lattice automorphisms. Naturally, the well known result [32, Theorem 3.2.10] on lattice homomorphisms between $C_0(K)$-spaces directly implies the structure of $\mathrm{Aut}^+(\mathbb{R}^n)$, but in this case, where $K = \{1, \dots, n\}$, this can be seen in an elementary fashion as below. Subsequently we determine the finite subgroups of $\mathrm{Aut}^+(\mathbb{R}^n)$. After that, we can describe the positive representations of a finite group in $\mathbb{R}^n$ and continue from there.

3.4.1 Description of $\mathbf{Aut}^+(\mathbb{R}^n)$

We denote the standard basis of $\mathbb{R}^n$ by $\{e_1, \dots, e_n\}$. A lattice automorphism must obviously map positive atoms to positive atoms, so each $T \in \mathrm{Aut}^+(\mathbb{R}^n)$ maps e_i to

$\lambda_{ji}e_j$ for some $\lambda_{ji} > 0$. This implies that T can be written uniquely as the product of a strictly positive multiplication (diagonal) operator and a permutation operator. We identify the group of permutation operators with S_n, so each $\sigma \in S_n$ corresponds to the operator mapping e_i to $e_{\sigma(i)}$. The group of strictly positive multiplication operators is identified with $(\mathbb{R}_{>0})^n$, and so there exist unique $m \in (\mathbb{R}_{>0})^n$ and $\sigma \in S_n$ such that $T = m\sigma$.

For $\sigma \in S_n$ and $m \in (\mathbb{R}_{>0})^n$, define $\sigma(m) \in (\mathbb{R}_{>0})^n$ by $\sigma(m)_i := m_{\sigma^{-1}(i)}$. This defines a homomorphism of S_n into the automorphism group of $(\mathbb{R}_{>0})^n$, hence we can form the corresponding semidirect product $(\mathbb{R}_{>0})^n \rtimes S_n$, with group operation $(m_1, \sigma_1)(m_2, \sigma_2) := (m_1\sigma_1(m_2), \sigma_1\sigma_2)$. On noting that, for $i = 1, \ldots, n$,

$$\sigma m \sigma^{-1} e_i = \sigma m e_{\sigma^{-1}(i)} = \sigma m_{\sigma^{-1}(i)} e_{\sigma^{-1}(i)} = m_{\sigma^{-1}(i)} e_i = \sigma(m)_i e_i = \sigma(m) e_i,$$

it follows easily that $\chi\colon (\mathbb{R}_{>0})^n \rtimes S_n \to \mathrm{Aut}^+(\mathbb{R}^n)$, defined by $\chi(m, \sigma) := m\sigma$, is a group isomorphism. From now on we identify $\mathrm{Aut}^+(\mathbb{R}^n)$ and $(\mathbb{R}_{>0})^n \rtimes S_n$, using either the operator notation or the semidirect product notation.

We let $p\colon \mathrm{Aut}^+(\mathbb{R}^n) \to S_n$, defined by $p(m, \sigma) := \sigma$, denote the canonical homomorphism of the semidirect product onto the second factor.

3.4.2 Description of the finite subgroups of $\mathrm{Aut}^+(\mathbb{R}^n)$

Let G be a finite subgroup of $\mathrm{Aut}^+(\mathbb{R}^n)$. Then $\ker(p|_G)$ can be identified with a finite subgroup of $\ker(p) = (\mathbb{R}_{>0})^n$. Clearly the only finite subgroup of $(\mathbb{R}_{>0})^n$ is trivial, and so $p|_G$ is an isomorphism. It follows that every finite subgroup of $\mathrm{Aut}^+(\mathbb{R}^n)$ is isomorphic to a finite subgroup of S_n. The next proposition makes this correspondence explicit.

Proposition 3.4.1. *Let A be the set of finite subgroups $G \subset \mathrm{Aut}^+(\mathbb{R}^n)$, and let B be the set of pairs (H, q), where $H \subset S_n$ is a finite subgroup and $q\colon H \to \mathrm{Aut}^+(\mathbb{R}^n)$ is a group homomorphism such that $p \circ q = \mathrm{id}_H$. Define $\alpha\colon A \to B$ and $\beta\colon B \to A$ by*

$$\alpha(G) := \left(p(G), (p|_G)^{-1}\right), \quad \beta(H, q) := q(H).$$

Then α and β are inverses of each other.

Proof. Clearly $p \circ (p|_G)^{-1} = \mathrm{id}_{p(G)}$, so α is well defined. Let $G \in A$, then

$$\beta(\alpha(G)) = \beta(p(G), (p|_G)^{-1}) = G.$$

Conversely, let $(H, q) \in B$, then $\alpha(\beta(H, q)) = \alpha(q(H)) = (p(q(H)), (p|_{q(H)})^{-1})$, and since $p \circ q = \mathrm{id}_H$, it follows that $p(q(H)) = H$ and that

$$(p|_{q(H)})^{-1} = (p|_{q(H)})^{-1} \circ p \circ q = q.$$

$\square$

By the above proposition each finite subgroup G of $\mathrm{Aut}^+(\mathbb{R}^n)$ is determined by a subgroup H of S_n and a homomorphism $q\colon H \to \mathrm{Aut}^+(\mathbb{R}^n)$ satisfying $p \circ q = \mathrm{id}_H$. We will now investigate such maps q. The condition $p \circ q = \mathrm{id}_H$ is equivalent with the existence of a map $f\colon H \to (\mathbb{R}_{>0})^n$, such that $q(\sigma) = (f(\sigma), \sigma)$ for $\sigma \in H$. For $\sigma, \tau \in H$, we have $q(\sigma\tau) = (f(\sigma\tau), \sigma\tau)$ and

$$q(\sigma)q(\tau) = (f(\sigma), \sigma)(f(\tau), \tau) = (\sigma(f(\tau))f(\sigma), \sigma\tau).$$

Hence q being a group homomorphism is equivalent with $f(\sigma\tau) = \sigma(f(\tau))f(\sigma)$ for all $\sigma, \tau \in H$, and such maps are called crossed homomorphisms.

Crossed homomorphisms of a finite group into a suitably nice abelian group $(A, +)$ (in our case $((\mathbb{R}_{>0})^n, \cdot)$) can be characterized by the following lemma, which states, in the language of group cohomology, that $H^1(H, A)$ is trivial.

Lemma 3.4.2. *Let H be a finite group acting on an abelian group $(A, +)$ such that, for all $a \in A$, there exists a unique element of H, denoted by $a/|H|$, satisfying $|H|(a/|H|) = a$. Let $f\colon H \to A$ be a map. Then f is a crossed homomorphism, i.e., $f(st) = s(f(t)) + f(s)$ for all $s, t \in H$, if and only if there exists an $a \in A$ such that*

$$f(s) = a - s(a) \quad \forall s \in H.$$

Proof. Suppose f is a crossed homomorphism. Let $a := \frac{1}{|H|}\sum_{r \in H} f(r)$, then, for $s \in H$,

$$\begin{aligned} s(a) &= \frac{1}{|H|}\sum_{r \in H} s(f(r)) \\ &= \frac{1}{|H|}\sum_{r \in H} [f(sr) - f(s)] \\ &= \frac{1}{|H|}\sum_{r \in H} [f(r) - f(s)] \\ &= a - f(s). \end{aligned}$$

Hence $f(s) = a - s(a)$, as required. The converse is trivial. $\square$

Combining this result with the previous discussion, we obtain the following.

Corollary 3.4.3. *Let H be a finite subgroup of S_n and let $q\colon H \to \mathrm{Aut}^+(\mathbb{R}^n)$ be a map. Then q is a homomorphism satisfying $p \circ q = \mathrm{id}_H$ if and only if there exists an $m \in (\mathbb{R}_{>0})^n$ such that*

$$q(\sigma) = (m\sigma(m)^{-1}, \sigma) \quad \forall \sigma \in H.$$

Rewriting this in multiplicative notation rather than semidirect product notation yields the following.

Corollary 3.4.4. *Let G be a finite subgroup of $\mathrm{Aut}^+(\mathbb{R}^n)$. Then there is a unique finite subgroup $H \subset S_n$ and an $m \in (\mathbb{R}_{>0})^n$ such that*

$$G = \left\{ m\sigma(m)^{-1}\sigma : \sigma \in H \right\} = mHm^{-1}.$$

Conversely, if $H \subset S_n$ is a finite subgroup and $m \in (\mathbb{R}_{>0})^n$, then $G \subset \mathrm{Aut}^+(\mathbb{R}^n)$ defined by the above equation is a finite subgroup of $\mathrm{Aut}^+(\mathbb{R}^n)$.

Proof. By Proposition 3.4.1, $G = q(p(G))$, for some $q\colon p(G) \to (\mathbb{R}_{>0})^n$ satisfying $p \circ q = \mathrm{id}_{p(G)}$. So $H = p(G)$ is unique, and the rest follows from the previous corollary. □

Note that, given a finite subgroup $G \subset \mathrm{Aut}^+(\mathbb{R}^n)$, the subgroup $H \subset S_n$ is unique, but the multiplication operator m in Corollary 3.4.4 is obviously not unique, e.g., both m and λm for $\lambda > 0$ induce the same G.

3.4.3 Positive finite dimensional Archimedean representations

In this subsection we obtain our main results on finite dimensional positive representations of finite groups in Archimedean spaces: explicit description of such representations (Theorem 3.4.5), decomposition into irreducible positive representations (Theorem 3.4.9) and description of irreducible positive representations up to order equivalence (Theorem 3.4.10).

Applying the results from the previous subsection, in particular Proposition 3.4.1 and Lemma 3.4.2, we obtain the following. Recall that we view $S_n \subset \mathrm{Aut}^+(\mathbb{R}^n)$, by identifying $\sigma \in S_n$ with a permutation matrix, and a representation $\pi\colon G \to S_n$ is called a *permutation representation*.

Theorem 3.4.5. *Let G be a finite group and $\rho\colon G \to \mathrm{Aut}^+(\mathbb{R}^n)$ a positive representation. Then there is a unique permutation representation π and an $m \in (\mathbb{R}_{>0})^n$ such that*

$$\rho_s = m\pi_s m^{-1} \quad \forall s \in G.$$

Conversely, any permutation representation $\pi\colon G \to S_n$ and $m \in (\mathbb{R}_{>0})^n$ define a positive representation ρ by the above equation.

Proof. Applying Proposition 3.4.1 to $\rho(G)$ and combining this with Lemma 3.4.2, $p\colon \rho(G) \to p\circ\rho(G)$ has an inverse of the form $q(\sigma) = m\sigma(m)^{-1}\sigma$ for some $m \in (\mathbb{R}_{>0})^n$ and all $\sigma \in p \circ \rho(G)$. We define $\pi := p \circ \rho$, then for $s \in G$,

$$\rho_s = (q \circ p)(\rho_s) = q(\pi_s) = m\pi_s(m)^{-1}\pi_s = m\pi_s m^{-1}.$$

This shows the existence of π. The uniqueness of π follows from the uniqueness of the factors in $(\mathbb{R}_{>0})^n$ and S_n in

$$\rho_s = m\pi_s m^{-1} = [m\pi_s(m)^{-1}]\pi_s.$$

The converse is clear. □

If ρ, π and m are related as in the above theorem, we will denote this as $\rho \sim (m, \pi)$. Note that, as in Corollary 3.4.4, π is unique, but m is not. Given the permutation representation π, m_1 and m_2 induce the same positive representations if and only if $m_1 m_2^{-1} = \pi_s(m_1 m_2^{-1})$ for all $s \in G$.

Recall that if $\rho, \theta \colon G \to \mathrm{Aut}^+(\mathbb{R}^n)$ are positive representations, we call them *order equivalent* if there exists an intertwiner $T \in \mathrm{Aut}^+(\mathbb{R}_{>0})^n$ between ρ and θ. We call them *permutation equivalent* if there exists an intertwiner $\sigma \in S_n$; this implies order equivalence.

Proposition 3.4.6. *Let G be a finite group and $\rho^1 \sim (m_1, \pi^1)$ and $\rho^2 \sim (m_2, \pi^2)$ be two positive representations of G in $\mathbb{R}^n$. Then ρ^1 and ρ^2 are order equivalent if and only if π^1 and π^2 are permutation equivalent.*

Proof. Suppose that ρ^1 and ρ^2 are order equivalent and let $T = (m, \sigma) \in \mathrm{Aut}^+(\mathbb{R}^n)$ be an intertwiner. Then, for all $s \in G$,

$$\rho_s^1 T = (m_1 \pi_s^1(m_1)^{-1}, \pi_s^1)(m, \sigma) = (m_1 \pi_s^1(m_1)^{-1} \pi_s^1(m), \pi_s^1 \sigma) \tag{3.4.1}$$
$$T \rho_s^2 = (m, \sigma)(m_2 \pi_s^2(m_2)^{-1}, \pi_s^2) = (m \sigma(m_2) \sigma \pi_s^2(m_2)^{-1}, \sigma \pi_s^2), \tag{3.4.2}$$

and since these are equal, σ is an intertwiner between π^1 and π^2.

Conversely, let σ be an intertwiner between π^1 and π^2. Then, by choosing $m = m_1 \sigma(m_2)^{-1}$ and $T = (m, \sigma) \in \mathrm{Aut}^+(\mathbb{R}^n)$, it is easily verified that, for all $s \in G$,

$$(m_1 \pi_s^1(m_1)^{-1} \pi_s^1(m), \pi_s^1 \sigma) = (m \sigma(m_2) \sigma \pi_s^2(m_2)^{-1}, \sigma \pi_s^2),$$

and so by (3.4.1) and (3.4.2), T intertwines ρ^1 and ρ^2. □

We immediately obtain that every positive representation is order equivalent to a permutation representation.

Corollary 3.4.7. *Let G be a finite group and let $\rho \sim (m, \pi)$ be a positive representation of G in $\mathbb{R}^n$. Then ρ is order equivalent to $(\mathbf{1}, \pi)$.*

Remark 3.4.8. The method in this subsection also yields a description of the homomorphisms from a finite group into a semidirect product $N \rtimes K$ with N torsion-free and $H^1(H, N)$ trivial for all finite subgroups $H \subset K$, but we are not aware of a reference for this fact.

Using Corollary 3.4.7, we obtain our decomposition theorem.

Theorem 3.4.9. *Let G be a finite group and $\rho \colon G \to \mathrm{Aut}^+(E)$ a positive representation in a nonzero finite dimensional Archimedean Riesz space E. Let $\{B_i\}_{i \in I}$ be the set of irreducible invariant bands in E. Then $E = \oplus_{i \in I} B_i$. Furthermore, any invariant band is a direct sum of B_i's.*

Proof. We may assume that $E = \mathbb{R}^n$, where bands are just linear spans of a number of standard basis vectors. By Corollary 3.4.7, we may assume that ρ is a permutation representation, which is induced by a group action on the set of basis elements. It is then clear that the irreducible invariant bands correspond to the orbits of this group action, and the invariant bands to unions of orbits. This immediately gives the decomposition of ρ into irreducible positive representations, and the description of the invariant bands. $\square$

We will now give a description of what could be called the finite dimensional Archimedean part of the order dual of a finite group. Note that Theorems 3.3.14 and Theorem 3.3.16 imply that a number of positive representations, irreducible in an appropriate way, are automatically in finite dimensional Archimedean spaces. Hence they fall within the scope of the next result, which is formulated in terms of a function lattice in order to emphasize the similarity with [43, Theorem III.10.4].

Theorem 3.4.10. *Let G be a finite group. If $H \subset G$ is a subgroup, let $(e_{tH})_{tH \in G/H}$ be the canonical basis for the finite dimensional Riesz space $C(G/H)$, defined by $e_{tH}(sH) = \delta_{tH,sH}$, for tH, $sH \in G/H$. Let $\pi^H \colon G \to \mathrm{Aut}^+(C(G/H))$ be the canonical positive representation corresponding to the action of G on G/H, so that $\pi^H_s e_{tH} := e_{stH}$ for $s, t \in G$. Then, whenever H_1 and H_2 are conjugate, π^{H_1} and π^{H_2} are order equivalent, and the map*

$$[H] \mapsto [\pi^H]$$

is a bijection between the conjugacy classes of subgroups of G and the order equivalence classes of irreducible positive representations of G in nonzero finite dimensional Archimedean Riesz spaces.

Proof. It follows easily from Theorem 3.2.2 that the map is well-defined. As a consequence of Corollary 3.4.7, every nonzero finite dimensional positive Archimedean representation is order equivalent to a permutation representation, arising from an action of G on $\{1, \ldots, n\}$ for some ≥ 1. Since the irreducibility of π is then equivalent to the transitivity of this group action, this shows that the map is surjective. As to injectivity, suppose that π^{H_1} and π^{H_2} are order equivalent, for subgroups H_1, H_2 of G. Let $n = |G : H_1| = |G : H_2|$, and consider $\mathbb{R}^n$ with standard basis $\{e_1, \ldots, e_n\}$. Choose a bijection between the canonical basis for $C(G/H_1)$ and $\{e_1, \ldots, e_n\}$, and likewise for the canonical basis of $C(G/H_2)$. This gives a lattice isomorphism between $C(G/H_1)$ and $\mathbb{R}^n$, and similarly for $C(G/H_2)$. After transport of structure, G has two positive representations on $\mathbb{R}^n$ which are order equivalent by assumption, and which originate from two permutation representations on the same set $\{1, \ldots, n\}$. As a consequence of Proposition 3.4.6, the permutation parts of these positive representations are permutation equivalent, i.e., the two G-spaces, consisting of $\{1, \ldots, n\}$ and the respective G-actions, are isomorphic G-spaces. Consequently, G/H_1 and G/H_2 are isomorphic G-spaces, and then Lemma 3.2.2 shows that H_1 and H_2 are conjugate. $\square$

If $n \in \mathbb{Z}_{\geq 0}$ and ρ is a positive representation, then $n\rho$ denotes the n-fold order direct sum of ρ. Combining the above theorem with Theorem 3.4.9, we immediately obtain the following, showing how the representations under consideration are built from canonical actions on function lattices on transitive G-spaces.

Corollary 3.4.11. *Let G be a finite group and let $H_1, \ldots, H_k$ be representatives of the conjugacy classes of subgroups of G. Let E be a finite dimensional Archimedean Riesz space and let $\rho\colon G \to \mathrm{Aut}^+(E)$ be a positive representation. Then, using the notation of Theorem 3.4.10, there exist unique $n_1, \ldots, n_k \in \mathbb{Z}_{\geq 0}$ such that ρ is order equivalent to*

$$n_1\pi^{H_1} \oplus \cdots \oplus n_k\pi^{H_k}.$$

3.4.4 Linear equivalence and order equivalence

If two unitary group representations are intertwined by a bounded invertible operator, they are also intertwined by a unitary operator [10, Section 2.2.2]. We will now investigate the corresponding natural question in the finite dimensional ordered setting: if two positive finite dimensional Archimedean representations are intertwined by an invertible linear map, are they order equivalent? By character theory, see for example [25, Theorem XVIII.2.3], two representations over the real numbers are linearly equivalent if and only if they have the same character. The following example, taken from the introduction of [26], therefore settles the matter.

Example 3.4.12. Let G be the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and conser the permutation representations $\pi^1, \pi^2\colon G \to \mathrm{Aut}^+(\mathbb{R}^6)$ determined by

$$\pi^1_{(0,1)} := (12)(34) \quad \pi^1_{(1,0)} := (13)(24) \quad \pi^2_{(0,1)} := (12)(34) \quad \pi^2_{(1,0)} := (12)(56).$$

Then, as is easily verified, π^1 and π^2 have the same character, and so they are linearly equivalent. However, they are not order equivalent, since by examining the orbits of standard basis elements it follows that the first representation splits into three irreducible positive representations of dimensions 1, 1, and 4, and the second splits into three irreducible positive representations of dimension 2 each.

Thus, in general, linear equivalence (equivalently: equality of characters) of positive representations does not imply order equivalence. One might then try to narrow down the field: is perhaps true that two *irreducible* positive representations, which are linearly equivalent, are order equivalent? In view of Theorem 3.4.10 and Theorem 3.2.2, this is asking whether a linear equivalence of the positive representations corresponding to two transitive G-spaces (which is equivalent to equality, for each group element, of the number of fixed points in the two spaces) implies that these G-spaces are isomorphic. The answer, again, is negative, but counterexamples are now more intricate to construct than above, and the reader is referred to [47, Theorem 1], providing an abundance of such counterexamples.

On the positive side, in some cases linear equivalence of irreducible positive representations does imply order equivalence, as shown by the next result.

Lemma 3.4.13. *Let G be a finite group, let N be a normal subgroup, and let $\pi^N\colon G \to \mathrm{Aut}^+(C(G/N))$ be the irreducible positive representations as in Theorem 3.4.10. Then an irreducible positive representation which is linearly equivalent with π^N is in fact order equivalent with π^N.*

Proof. Passing to an order equivalent model we may, in view of Theorem 3.4.10, assume that the other irreducible positive representation is π^H, for a subgroup H of G. The fact that G/N is a group implies easily that the character of π^N equals $|G:N|\mathbf{1}_N$. Since the character of π^H, which is equal to that of π^N by their linear equivalence, is certainly nonzero on H, we see that $H \subset N$. On the other hand, equality of dimensions yields $|G:N| = |G:H|$, hence $|N| = |H|$. We conclude that $H = N$. □

Combining Theorem 3.4.10 with the previous lemma yields the following.

Corollary 3.4.14. *Let G be a finite group with only normal subgroups. If two finite dimensional irreducible positive representations of G are linearly equivalent (equivalently: have the same character), they are order equivalent.*

Thus, for such groups (so-called Dedekind groups), and in particular for abelian groups, the classical correspondence between characters and irreducible representations survives in an ordered context—where, naturally, "irreducible" has a different meaning. However, as Example 3.4.12 shows, already for abelian groups this correspondence breaks down for reducible positive representations.

3.5 Induction

In this section we will examine the theory of induction in the ordered setting from a categorical point of view. It turns out that the results are to a large extent analogous to the linear case as covered in many sources, e.g., [25, Section XVIII.7]. Still, it seems worthwhile to go through the motions, as a preparation for future more analytical constructions, and in doing so we then also obtain a slight bonus (the original ordered module is embedded in the induced one as a sublattice, even though this was not required), keep track of several notions of irreducibility, and also observe that Frobenius reciprocity holds only partially. Since we do not consider topological issues at the moment, we are mostly interested in the case where the group is finite, but the theory is developed at little extra cost in general for arbitrary groups and subgroups. Our approach is thus slightly more general than, e.g., the approach in [25], as we do not require our groups to be finite or our subgroups to be of finite index.

For the rest of the section, G is a not necessarily finite group, H is a subgroup of G, not necessarily finite or of finite index, R is a system of representatives of G/H, and the Riesz spaces are not assumed to be finite dimensional. The only finiteness condition is in Corollary 3.5.11, where G is assumed to be finite.

3.5.1 Definitions and basic properties

A pair (E, ρ), where E is a Riesz space and $\rho\colon G \to \mathrm{Aut}^+(E)$ is a positive representation, is called an *ordered G-module*. In this notation, we will often omit the representation ρ. If E is an ordered G-module, it is also an ordered H-module by restricting the representation to H. If (E,ρ) and (F,θ) are ordered G-modules, then the positive cone of positive intertwiners between ρ and θ will be denoted by $\mathrm{Hom}_G^+(E,F)$. Two ordered G-modules are *isomorphic ordered G-modules* if there exists an intertwining lattice isomorphism.

Definition 3.5.1. Let F be an ordered H-module. A pair $(\mathrm{Ind}_H^G(F), j)$, where $\mathrm{Ind}_H^G(F)$ is an ordered G-module and $j \in \mathrm{Hom}_H^+(F, \mathrm{Ind}_H^G(F))$ is called an *induced ordered module* of F from H to G if it satisfies the following universal property:

For any ordered G-module E and $T \in \mathrm{Hom}_H^+(F,E)$, there is a unique map $\overline{T} \in \mathrm{Hom}_G^+(\mathrm{Ind}_H^G(F), E)$ such that $T = \overline{T} \circ j$, i.e., such that the following diagram is commutative:

$$\begin{tikzcd} F \arrow[rr, "T"] \arrow[dr, "j"'] & & E \\ & \mathrm{Ind}_H^G(F) \arrow[ur, "\overline{T}"'] & \end{tikzcd}$$

If θ is the positive representation of H turning F into an ordered H-module, then the positive representation of G turning $\mathrm{Ind}_H^G(F)$ into an ordered G-module will be denoted by $\mathrm{Ind}_H^G(\theta)$ and will be called the *induced positive representation* of θ from H to G.

First we will show, by the usual argument, that the induced ordered module is unique up to isomorphism of ordered G-modules.

Lemma 3.5.2. *Let F be an ordered H-module and let (E_1, j_1) and (E_2, j_2) be induced ordered modules of F from H to G. Then E_1 and E_2 are isomorphic as ordered G-modules.*

Proof. Using the universal property, we obtain the unique maps $\overline{j_1} \in \mathrm{Hom}_G^+(E_2, E_1)$ satisfying $j_1 = \overline{j_1} \circ j_2$ and $\overline{j_2} \in \mathrm{Hom}_G^+(E_1, E_2)$ satisfying $j_2 = \overline{j_2} \circ j_1$. It follows that

$$j_1 = \overline{j_1} \circ j_2 = \overline{j_1} \circ \overline{j_2} \circ j_1.$$

Now consider the ordered G-module E_1, and apply its universal property to itself - we obtain the *unique* map $\mathrm{id}_{E_1} \in \mathrm{Hom}_G^+(E_1, E_1)$ such that $j_1 = \mathrm{id}_{E_1} \circ j_1$. Together with the above equation, this shows that $\overline{j_1} \circ \overline{j_2} = \mathrm{id}_{E_1}$. Similarly we obtain $\overline{j_2} \circ \overline{j_1} = \mathrm{id}_{E_2}$, and so $\overline{j_2}$ is an isomorphism of ordered G-modules. $\square$

We will now construct the induced ordered module, which is the usual induced module, but now with an additional lattice structure. Let (F,θ) be an ordered H-module. We define the ordered vector space

$$\tilde{E} := \{f : G \to F |\ f(st) = \theta_{t^{-1}} f(s)\ \forall s \in G,\ \forall t \in H\}, \tag{3.5.1}$$

with pointwise ordering. Using that $\theta_{t^{-1}}$ is a lattice isomorphism for $t \in H$, one easily verifies that $\tilde{E}$ is a Riesz space, with pointwise lattice operations. Furthermore, if $S \subset G$ is a subset, then we define the subset

$$E_S := \{f \in \tilde{E} \mid \mathrm{supp}\,(f) \subset S\}. \tag{3.5.2}$$

Let $\rho : G \to \mathrm{Aut}^+(\tilde{E})$ be defined by $(\rho_s f)(u) := f(s^{-1}u)$, for $s, u \in G$. Then ρ is a positive representation turning $\tilde{E}$ into an ordered G-module. Moreover, for $s \in G$, $\mathrm{supp}\,(\rho_s f) = s \cdot \mathrm{supp}\,(f)$, so $\rho_s E_H = E_{sH}$. Now we define the ρ-invariant Riesz subspace

$$E := \bigoplus_{r \in R} E_{rH} = \bigoplus_{r \in R} \rho_r E_H \subset \tilde{E}. \tag{3.5.3}$$

For $x \in F$, define a function $j(x)\colon G \to F$ by $j(x)(t) := \theta_{t^{-1}} x$ for $t \in H$, and $j(x)(t) := 0$ for $t \notin H$. It is routine to verify that $j(x) \in E_H$ for all $x \in F$, and that $j\colon F \to E_H$ is an isomorphism of ordered H-modules between (F, θ) and (E_H, ρ). This last fact and the fact that $E = \oplus_{r \in R} \rho_r F$ as an order direct sum will be used to prove that (E, j) actually satisfies the desired universal property.

Lemma 3.5.3. *Let (E, ρ) be an ordered G-module and let F be a Riesz subspace of E which is invariant under the restricted representation $\theta = \rho|_H$ of H in E, and such that $E = \oplus_{r \in R} \rho_r F$ as an order direct sum. Let $j\colon F \to E$ be the embedding. Then (E, j) is an induced ordered module of F from H to G.*

Proof. Let (E', ρ') be another ordered G-module, and let $T : F \to E'$ be a positive linear map such that $T(\theta_t x) = \rho'_t T(x)$ for all $t \in H$ and $x \in F$. We have to show that there exists a unique positive linear map $\overline{T} : E \to E'$ extending T and satisfying $\overline{T} \circ \rho_s = \rho'_s \circ \overline{T}$ for all $s \in G$.

We follow the proof of [45, Lemma 3.3.1]. If $\overline{T}$ satisfies these conditions, and if $x \in \rho_r F$ for $r \in R$, then $\rho_r^{-1} x \in F$, and so

$$\overline{T}(x) = \overline{T}(\rho_r \rho_r^{-1} x) = \rho'_r \overline{T}(\rho_r^{-1} x) = \rho'_r T(\rho_r^{-1} x).$$

This formula determines $\overline{T}$ on $\rho_r F$, hence on E since it is the direct sum of the $\rho_r F$. This proves the uniqueness of $\overline{T}$.

For the existence, let $x \in \rho_r F$, then we define $\overline{T}(x) := \rho'_r T(\rho_r^{-1} x)$ as above. This does not depend on the choice of representative r, since if we replace r by rt with $t \in H$, we have

$$\rho'_{rt} T(\rho_{rt}^{-1} x) = \rho'_r \rho'_t T(\theta_t^{-1} \rho_r^{-1} x) = \rho'_r T(\theta_t \theta_t^{-1} \rho_r^{-1} x) = \rho'_r T(\rho_r^{-1} x).$$

Since E is the direct sum of the $\rho_r F$, there exists a unique linear map $\overline{T}\colon E \to E'$ which extends the partial mappings thus defined on the $\rho_r F$. It is easily verified that $\overline{T} \rho_s = \rho'_s T$ for all $s \in G$. Since all mappings involved are positive, $\overline{T}$ is positive as well. $\square$

Corollary 3.5.4. *Let F be an ordered H-module. Then the induced ordered module $(\mathrm{Ind}_H^G(F), j)$ of F from H to G exists, and $\mathrm{Ind}_H^G(F)$ is unique up to isomorphism of ordered G-modules. The positive map j is actually an injective lattice homomorphism of the ordered H-module F into the ordered H-module $\mathrm{Ind}_H^G(F)$. Moreover, if E is finite dimensional and H has finite index, then* $\dim(\mathrm{Ind}_H^G(F)) = |G : H| \dim(E)$.

Proof. The existence follows from Lemma 3.5.3 and the construction preceding it, which also shows that j has the property as described. The uniqueness follows from Lemma 3.5.2. The last statement follows from (3.5.3). $\square$

We continue with some properties of the induced positive representation.

Proposition 3.5.5. *Let θ be a positive representation of a subgroup $H \subset G$. If $\mathrm{Ind}_H^G(\theta)$ is either band irreducible, or ideal irreducible, or projection band irreducible, then so is θ.*

Proof. We will prove this for ideals, the other cases are identical. Let E be as in (3.5.1), (3.5.2) and (3.5.3). Suppose θ is not ideal irreducible, so there exists a proper nontrivial θ-invariant ideal $B \subset E_\theta$. Then $\{f \in E : f(G) \subset B\} \subset E$ is a proper nontrivial $\mathrm{Ind}_H^G(\theta)$-invariant ideal, so $\mathrm{Ind}_H^G(\theta)$ is not ideal irreducible. $\square$

Proposition 3.5.6 (Induction in stages)**.** *Let $H \subset K \subset G$ be a chain of subgroups of G, and let F be an ordered H-module. Then*

$$(\mathrm{Ind}_K^G(\mathrm{Ind}_H^K(F)), j_K^G \circ j_H^K)$$

is an induced ordered module of F from H to G.

Proof. Let E be an ordered G-module. Consider the following diagram:

$$\begin{tikzcd}
F \arrow[r, "j_H^K"] \arrow[dr, "T"'] & \mathrm{Ind}_H^K(F) \arrow[r, "j_K^G"] \arrow[d, "\overline{T}"] & \mathrm{Ind}_K^G(\mathrm{Ind}_H^K(F)) \arrow[dl, "\overline{\overline{T}}"] \\
& E &
\end{tikzcd}$$

Here $\overline{T}$ is the unique positive map generated by T, and $\overline{\overline{T}}$ is the unique positive map generated by $\overline{T}$. Since the diagram is commutative, $T = \overline{\overline{T}} \circ (j_K^G \circ j_H^K)$. If S is another positive map satisfying $T = S \circ (j_K^G \circ j_H^K)$, then $(S \circ j_K^G) \circ j_H^K = T = \overline{T} \circ j_H^K$, and so $S \circ j_K^G = \overline{T}$ by the uniqueness of $\overline{T}$. This in turn implies that $S = \overline{\overline{T}}$ by the uniqueness of $\overline{\overline{T}}$, and so $(\mathrm{Ind}_K^G(\mathrm{Ind}_H^K(F)), j_K^G \circ j_H^K)$ satisfies the universal property, as desired. $\square$

3.5.2 Frobenius reciprocity

This subsection is concerned with the implications, or rather their absence, of the functorial formulation of Frobenius reciprocity for multiplicities of irreducible positive representations in induced ordered modules.

We start with the usual consequence of the categorical definition of the induced ordered module: induction from H to G is the left adjoint functor of restriction from G to H, for an arbitrary group G and subgroup H.

Proposition 3.5.7 (Frobenius Reciprocity)**.** *Let F be an ordered H-module and let E be an ordered G-module. Then there is a natural isomorphism of positive cones*

$$\mathrm{Hom}_H^+(F, E) \cong \mathrm{Hom}_G^+(\mathrm{Ind}_H^G(F), E).$$

Proof. The existence of the natural bijection is an immediate consequence of the very definition of the induced module in Definition 3.5.1. For $T, S \in \mathrm{Hom}_H^+(F, E)$ and $\lambda, \mu \geq 0$ we have $\overline{\lambda T + \mu S} = \lambda\overline{T} + \mu\overline{S}$ as a consequence of the uniqueness part of Definition 3.5.1 and the positivity of $\lambda\overline{T} + \mu\overline{S}$. Hence the two positive cones are isomorphic. □

Now suppose G is a finite group. For finite dimensional positive Archimedean representations of G we have a unique decomposition into irreducible positive representations, according to Corollary 3.4.11. Hence the notion of multiplicity is available, and if ρ_1 is a finite dimensional positive representation and ρ_2 a finite dimensional irreducible positive representation of G, we let $m(\rho_1, \rho_2)$ denote the number of times that a lattice isomorphic copy of ρ_1 occurs in the decomposition of ρ_2 of Corollary 3.4.11. Now Let θ be a finite dimensional irreducible positive representation of a subgroup $H \subset G$ and let ρ be a finite dimensional irreducible positive representation of G. In view of the purely linear theory, cf. [45, Proposition 21], and its generalization to unitary representations of compact groups, cf. [52, Theorem 5.9.2], it is natural to ask whether

$$m(\rho, \mathrm{Ind}_H^G(\theta)) = m(\theta, \rho|_H)$$

still holds in our ordered context. In the linear theory, and also for compact groups, this follows from the fact that the dimensions of spaces of intertwining operators in the analogues of Proposition 3.5.7 can be interpreted as a multiplicities. Since the sets in Proposition 3.5.7 are cones and not vector spaces, and their elements are not even necessarily lattice homomorphisms, there seems little chance of success in our case. Indeed, Frobenius reciprocity in terms of multiplicities does not hold for ordered modules, as is shown by the following counterexample. We let $\theta\colon H = \{e\} \to \mathrm{Aut}^+(\mathbb{R})$ be the trivial representation; then $\mathrm{Ind}_H^G(\theta)$ is the left regular representation of G on the Riesz space with atomic basis $\{e_s\}_{s\in G}$. This set of basis elements has only one G-orbit, hence $\mathrm{Ind}_H^G(\theta)$ is band irreducible. Therefore, if ρ is an arbitrary irreducible positive representation of G, $m(\rho, \mathrm{Ind}_H^G(\theta))$ is at most one. On the other hand, $\rho|_H$ decomposes as $\dim\rho$ copies of the trivial representation θ, so $m(\theta, \rho|_H) = \dim\rho$.

Another counterexample, where H is nontrivial, can be obtained by taking $G = \mathbb{Z}/4\mathbb{Z}$ and $H = \{0, 2\}$, and taking θ and ρ to be the left regular representation of H and G, respectively. Then these are irreducible positive representations of the respective groups, and it can be verified that $\mathrm{Ind}_H^G(\theta) \cong \rho$, so $m(\rho, \mathrm{Ind}_H^G(\theta)) = 1$, but $\rho|_H \cong \theta \oplus \theta$, so $m(\theta, \rho|_H) = 2$.

3.5.3 Systems of imprimitivity

In this final subsection, we consider systems of imprimitivity in the ordered setting. As before, G is an arbitrary group and $H \subset G$ an arbitrary subgroup. We start with an elementary lemma, which is easily verified.

Lemma 3.5.8. *Let E and F be Riesz spaces and let $T\colon E \to F$ be a lattice isomorphism. If $B \subset E$ is a projection band in E, then TB is a projection band in F, and the corresponding band projections are related by $P_{TB} = TP_BT^{-1}$.*

Let $\rho\colon G \to \mathrm{Aut}^+(E)$ be a positive representation. Suppose there exists a G-space Γ, and a family of Riesz subspaces $\{E_\gamma\}_{\gamma\in\Gamma}$, such that $E = \oplus_{\gamma\in\Gamma}E_\gamma$ as an order direct sum and $\rho_s E_\gamma = E_{s\gamma}$ for all $s \in G$. Then we call the family $\{E_\gamma\}_{\gamma\in\Gamma}$ an *ordered system of imprimitivity* for ρ. If $A \subset \Gamma$, then the order decomposition

$$E = \left(\bigoplus_{\gamma\in A} E_\gamma\right) \oplus \left(\bigoplus_{\gamma\in A^c} E_\gamma\right)$$

implies that $\oplus_{\gamma\in A}E_\gamma$ is a projection band. In particular, E_γ is a projection band for all $\gamma \in \Gamma$. For a subset $A \subset \Gamma$, let $P(A)$ denote the band projection $P_{\oplus_{\gamma\in A}E_\gamma}$ onto $\oplus_{\gamma\in A}E_\gamma$. Then the assignment $A \mapsto P(A)$ is a band projection valued map satisfying

$$P\left(\bigcup_{i\in I} A_i\right) x = \sup_{i\in I} P(A_i)x \tag{3.5.4}$$

for all $x \in E^+$ and all collections of subsets $\{A_i\}_{i\in I} \subset \Gamma$. An equivalent formulation of (3.5.4) is $P(\sup_i A_i) = \sup_i P(A_i)$, where the first supremum is taken in the partially ordered set of subsets of Γ, and the second supremum is taken in the partially ordered space of regular operators on E. Either formulation is the ordered analogue of a strongly σ-additive spectral measure. Furthermore, the map P is covariant in the sense that, for $s \in G$,

$$P(sA) = P_{\oplus_{\gamma\in A}E_{s\gamma}} = P_{\rho_s\oplus_{\gamma\in A}E_\gamma} = \rho_s\left(P_{\oplus_{\gamma\in A}E_\gamma}\right)\rho_s^{-1} = \rho_s P(A)\rho_s^{-1},$$

where the above lemma is used in the penultimate step. Every positive representation admits a system of imprimitivity where Γ has exactly one element, and such a system of imprimitivity will be called trivial. A system of imprimitivity is called transitive if the action of G on Γ is transitive.

Definition 3.5.9. A positive representation ρ is called *primitive* if it admits only the trivial ordered system of imprimitivity.

Every decomposition of E into ρ-invariant projection bands corresponds to an ordered system of imprimitivity where the action of G on Γ is trivial, so ρ is projection band irreducible if and only if ρ admits no nontrivial ordered system of imprimitivity with a trivial action. Hence a primitive positive representation is projection band irreducible, i.e., order indecomposable.

Theorem 3.5.10 (Imprimitivity Theorem)**.** *Let ρ be a positive representation of G. The following are equivalent:*

(*i*) *ρ admits a nontrivial ordered transitive system of imprimitivity;*

(*ii*) *There exists a proper subgroup $H \subset G$ and a positive representation θ of H such that ρ is order equivalent to $\mathrm{Ind}_H^G(\theta)$.*

Proof. $(ii) \Rightarrow (i)$: Suppose that ρ is order equivalent to $\mathrm{Ind}_H^G(\theta)$. Let Γ be the transitive G-space G/H, which is nontrivial because H is proper, and consider the spaces E and $\{E_{sH}\}_{sH\in\Gamma}$ defined in (3.5.1), (3.5.2) and (3.5.3). By the discussion following these definitions, $E = \oplus_{sH\in\Gamma} E_{sH}$, and $\rho_u E_{sH} = E_{usH}$, so this defines a nontrivial transitive system of imprimitivity.

$(i) \Rightarrow (ii)$: Suppose ρ admits a nontrivial transitive ordered system of imprimitivity. Then by Theorem 3.2.2 we may assume $\Gamma = G/H$ for some subgroup H, which must be proper since Γ is nontrivial. Choose a system of representatives R of G/H, then we may assume $\Gamma = R$. Assume that the representative of H in R is e. We have that $E = \oplus_{r\in R} E_r$, and if $t \in H$, then t acts trivially on $e \in R$, so $\rho_t E_e = E_e$. Therefore we can define $\theta\colon H \to \mathrm{Aut}^+(E_e)$ by restricting ρ to H and letting it act on E_e. Then by the definition of the system of imprimitivity $\{E_r\}_{r\in R}$ we have $\rho_r E_e = E_r$, and so

$$\bigoplus_{r\in R} \rho_r E_e = \bigoplus_{r\in R} E_r = E,$$

which implies by Lemma 3.5.3 that ρ is induced by θ. □

Corollary 3.5.11. *All projection band irreducible positive representations of a finite group G are induced by primitive positive representations.*

Proof. Let ρ be a projection band irreducible representation. If ρ is primitive we are done, so assume it is not primitive. Then there exists a nontrivial ordered system of imprimitivity $\{E_\gamma\}_{\gamma\in\Gamma}$. Then for each orbit in Γ, the direct sum of E_γ, where γ runs through the orbit, is a ρ-invariant projection band. Since ρ is projection band irreducible, this implies that Γ is transitive. Therefore, by the Imprimitivity Theorem 3.5.10, ρ is induced by a positive representation of a proper subgroup of G, which is projection band irreducible by Proposition 3.5.5. If this representation is primitive we are done, and if not we keep repeating the process until a representation is induced by a primitive positive representation. Then by Proposition 3.5.6 the representation ρ is induced by this primitive positive representation as well. □

We note that, in the above corollary, the representations need not be finite dimensional, and that it trivially implies a similar statement for band irreducible and ideal irreducible positive representations of a finite group.

Acknowledgements

The authors thank Lenny Taelman for helpful comments and suggestions.

Chapter 4

Compact groups of positive operators on Banach lattices

This chapter has been submitted for publication as: M. de Jeu and M. Wortel, "Compact groups of positive operators on Banach lattices". It is available as arXiv:1112.1611.

Abstract. In this paper we study groups of positive operators on Banach lattices. If a certain factorization property holds for the elements of such a group, the group has a homomorphic image in the isometric positive operators which has the same invariant ideals as the original group. If the group is compact in the strong operator topology, it equals a group of isometric positive operators conjugated by a single central lattice automorphism, provided an additional technical assumption is satisfied, for which we again have only examples. We obtain a characterization of positive representations of a group with compact image in the strong operator topology, and use this for normalized symmetric Banach sequence spaces to prove an ordered version of the decomposition theorem for unitary representations of compact groups. Applications concerning spaces of continuous functions are also considered.

4.1 Introduction and overview

In this paper we continue our efforts, initiated in Chapter 3, to develop a theory of strongly continuous positive representations of locally compact groups in Banach lattices. In Chapter 3 we investigated positive representations of finite groups. We showed that a principal band irreducible positive representation of a finite groups in a Riesz space is finite dimensional, and that the representation space is necessarily Archimedean. Furthermore, we classified such irreducible representation and showed that each Archimedean finite dimensional positive representations is an order direct sum of irreducible positive representations. Here we go a step further and consider strongly continuous positive representations of compact groups in Banach lattices.

Given the relative ease with which unitary representations of compact groups can be treated, this is the natural step to take and one would like to achieve a better understanding of issues related to irreducibility and decomposition in this context. Since the image of such a representation is a group of positive operators, we examine groups of positive operators, and since the image is compact in the strong operator topology, we are especially interested in compact (in the strong operator topology) groups of positive operators. In fact, most of the work in this paper is aimed at a better understanding of such compact groups. Once this is achieved, the transition to strongly continuous positive representations with compact image is not complicated.

There are not too many papers on groups of positive operators. In [11], uniformly bounded groups of positive operators on $C_c(\Omega)$ and $C_0(\Omega)$ are investigated in detail, where Ω is a locally compact Hausdorff space. These groups are studied using group actions on the underlying space Ω and group cohomology methods. Amongst others, it is shown in [11, Example 4.1] that a strongly continuous positive representation of a compact group on $C_0(\Omega)$ equals an isometric strongly continuous representation conjugated by a single central lattice automorphism, a result which we obtain as a special case of a more general statement, cf. Theorem 4.4.1 below.

In the case where the group G is compact in the strong operator topology, which is the main focus of our paper, a basic result is [43, Theorem III.10.4]. It was published in [33], which in turn is based on unpublished lecture notes by H.P. Lotz. It gives information concerning the structure of G as well as the Banach lattice G acts on, under the additional assumption that the action has only trivial invariant closed ideals. Amongst others, it states that the pertinent lattice can be found between $C(G/H)$ and $L^1(G/H)$, for some closed subgroup H of G, and the group acts as the group of left quasi-rotations induced by the natural action of G on G/H.

By studying the spectrum of lattice homomorphisms, [44] also contains some results about groups of positive operators, in particular it is shown in [44, Corollary 3.10] that a uniformly bounded group of positive operators on a Banach lattice is discrete in the norm topology, a result we obtain in the special case of groups which are compact in the strong operator topology on certain sequence spaces, cf. Corollary 4.5.6 below.

Beyond these results not much seems to be known. Naturally, there is a theory of one-parameter semigroups of positive operators, see, e.g., [3], but we are not aware of issues of irreducibility or decomposition into irreducibles being considered in detail for such semigroups.

In this paper we study groups of positive operators, or equivalently, groups of lattice automorphisms, with the property that every element can be written as a product of a central lattice automorphism and an isometric lattice automorphism. Remarkably enough, in the Banach lattices we consider in this paper, *every* lattice automorphism is such a product. However, there are Banach lattices for which this fails, cf. Example 4.3.1. The Banach lattices for which this holds true, as shown in this paper, include the normalized symmetric Banach sequence spaces (Section 4.5) and spaces of continuous functions (Section 4.6). Moreover, in these spaces we have a concrete description of both the central lattice automorphisms and the isometric lattice automorphisms. In the general case, for all groups with the aforementioned factorization property, we show that there is a group of isometric lattice automorphisms with the same invariant ideals as the original group, cf. Theorem 4.3.2. This is applied to the Banach sequence spaces mentioned above, where the isometric lattice automorphisms are easily described as permutations operators, and without too much effort one thus obtains a decomposition of a positive representation of an arbitrary group in such a Banach sequence space into band irreducibles, cf. Theorem 4.5.7. This result is reminiscent of the familiar decomposition theorem for strongly continuous unitary representations of compact groups into finite dimensional irreducible representations, but here the representation need not be strongly continuous, the group need not be compact, and the (order) irreducibles can be infinite dimensional.

Suppose the original group of automorphisms with the above factorization property is compact in the strong operator topology. As a first thought, we can equip the Banach lattice E with an equivalent lattice norm $|||\cdot|||$, defined by

$$|||x||| := \int_G \|Tx\| \, dT \quad \forall x \in E,$$

where dT denotes the Haar measure on the compact group G. Under this norm, the group G is now easily seen to be a group of isometric lattice automorphisms. However, by changing the norm, the isometries change as well, and any nice description of the original isometries need not survive this transformation. Hence this does not seem useful. Instead, we impose an additional technical assumption (Assumption 4.3.3) on the Banach lattice. Under this assumption, which we show to hold for normalized symmetric Banach sequence spaces with order continuous norm and spaces of continuous function, we can actually show that such a compact group is isomorphic as a topological group with the aforementioned group of isometric lattice automorphisms with the same invariant ideals. Moreover, we can characterize such groups G: they are precisely the groups of the form $G = mHm^{-1}$, for a unique compact group H of isometric lattice automorphisms, and a (non-unique) central lattice automorphism m, cf. Theorem 4.3.8. This is especially useful whenever we have a nice description of the central lattice automorphisms and the isometric lattice automorphisms, as in the spaces mentioned above. Along the same lines, one can show that positive representations with compact image in such spaces are precisely the conjugates of isometric representations, cf. Theorem 4.4.1. Moreover, in the case that we have a positive representation in a normalized symmetric Banach sequence space with order continuous norm or in ℓ^∞ as in Theorem 4.5.7, and the positive representation has compact image, the irreducible bands are finite dimensional, so that the the analogy with unitary representations of compact groups is then complete. For positive representations with compact image in spaces of continuous function, one cannot in general obtain such a direct sum type decomposition as in Theorem 4.5.7, and further research is necessary to see whether there is still a structure theorem for such representations in terms of band irreducible ones. As a preparation, we include a number of results on the invariant closed ideals, bands and projection bands for these representations.

The structure of this paper is as follows.

In Section 4.2 we introduce the basic notation and terminology. After establishing a few facts on groups of invertible operators and representations, we give a new proof of the fact that the center of a Banach lattice is isometrically algebra and lattice isomorphic to $C(K)$, for some compact Hausdorff space K. We also obtain some results on integrating strongly continuous center valued functions. In Section 4.3 we consider groups of lattice automorphisms for which every element is the product of a central lattice automorphism and an isometric lattice automorphism. We immediately obtain that there exists a group of isometric lattice automorphisms having the same invariant ideals as the original group. Then we state the technical Assumption 4.3.3, and under this assumption we are able to show one of our main results, Theorem 4.3.8, which states that every group of lattice automorphisms with this factorization property, and which is compact in the strong operator topology, equals a group of isometric lattice automorphisms conjugated by a central lattice automorphism. Using similar ideas, it is shown in Section 4.4 that positive representations with compact (in the strong operator topology) image are isometric positive representations conjugated with a central lattice automorphism. We then show that two positive representations with compact image are order equivalent if and only if their isometric parts are (isometrically) order equivalent. In Section 4.5 we define and examine normalized symmetric Banach sequence spaces. We show that all lattice automorphisms on such spaces are a product of a central lattice automorphism and an isometric lattice automorphism, and that, if the space has order continuous norm, the technical Assumption 4.3.3 holds. Then we apply the results from Section 4.3 and Section 4.4 to characterize compact groups of lattice automorphisms and positive representations with compact image. We also obtain Theorem 4.5.7, the aforementioned ordered version of the decomposition theorem for unitary representations of compact groups. Finally, in Section 4.6, we examine the Banach lattice $C_0(\Omega)$ for locally compact Hausdorff spaces Ω. Again we show that all lattice automorphisms are a product of a central lattice automorphism and an isometric lattice automorphism, and that Assumption 4.3.3 holds, and we apply the results from Section 4.3 and Section 4.4 to characterize compact groups of lattice automorphisms and positive representations with compact image. We finish with Proposition 4.6.7, which characterizes invariant closed ideals, bands and projection bands of positive representations with compact image.

4.2 Preliminaries

In this section we discuss various facts concerning the strong operator topology, groups of invertible operators, positive representations, the center of a Banach lattice, and integration of strongly continuous center valued functions.

If X is a Banach space, then $\mathcal{L}(X)$ denotes the bounded operators on X, and this space equipped with the strong operator topology will be denoted by $\mathcal{L}_s(X)$. Subsets of $\mathcal{L}_s(X)$ are always assumed to be equipped with the strong operator topology. It follows from the principle of uniform boundedness that compact subsets of $\mathcal{L}_s(X)$

are uniformly bounded. In this topology multiplication is separately continuous, and the multiplication is simultaneously continuous when the first variable is restricted to uniformly bounded subsets. The next lemma is concerned with the continuity of the inverse.

Lemma 4.2.1. *Let X be a Banach space and $H \subset \mathcal{L}_s(X)$ be a set of invertible operators such that H^{-1} is uniformly bounded. Then taking the inverse in H is continuous.*

Proof. Let $M > 0$ satisfy $\|T^{-1}\| \leq M$ for all $T \in H$, and let (T_i) be a net in H that converges strongly to $T \in H$. Let $x \in X$, then $x = Ty$ for some $y \in X$, and by the strong convergence of T_i to T,

$$\left\|T_i^{-1}x - T^{-1}x\right\| = \left\|T_i^{-1}(Ty - T_iy)\right\| \leq M\,\|Ty - T_iy\| \to 0.$$

□

Corollary 4.2.2. *Let X be a Banach space and let $H \subset \mathcal{L}_s(X)$ be a compact set and a group of invertible operators. Then H is a compact topological group.*

Proof. Compact subsets of $\mathcal{L}_s(X)$ are uniformly bounded, so the corollary follows from Lemma 4.2.1 and the simultaneous continuity of multiplication on uniformly bounded subsets of $\mathcal{L}_s(X)$. □

As a consequence, the group H in Corollary 4.2.2 has an invariant measure, a fact which will be instrumental in the proof of the key Lemma 4.3.7 below.

We continue with another lemma involving the strong operator topology, to be used in Lemma 4.3.5.

Lemma 4.2.3. *Let X be a Banach space, and let $A, B \subset \mathcal{L}_s(X)$ be equipped with the strong operator topology, such that $T_1S_1 = T_2S_2$ if and only if $T_1 = T_2$ and $S_1 = S_2$, for all $T_1, T_2 \in A$ and $S_1, S_2 \in B$. Define $p_A\colon A \cdot B \to A$ and $p_B\colon A \cdot B \to B$ by $p_A(TS) := T$ and $p_B(TS) = S$, for $TS \in A \cdot B$. Let $C \subset A \cdot B$ be a subset such that $p_A(C)$ is uniformly bounded and all elements of $p_B(C)$ are surjective. Then, if p_B restricted to C is continuous, p_A restricted to C is continuous as well.*

Proof. Let $M > 0$ satisfy $\|T\| \leq M$ for all $T \in A$ and $S \in B$ with $TS \in C$. Suppose p_B restricted to C is continuous, and let (T_iS_i) be a net in C that converges strongly to $TS \in C$, where (T_i) is a net in A and $T \in A$, and (S_i) is a net in B and $S \in B$. Let $x \in X$, then $x = Sy$ for some $y \in X$, and

$$\begin{aligned}
\|T_ix - Tx\| &= \|T_iSy - TSy\| \\
&\leq \|T_iSy - T_iS_iy\| + \|T_iS_iy - TSy\| \\
&\leq M\,\|Sy - S_iy\| + \|T_iS_iy - TSy\|\,,
\end{aligned}$$

which converges to zero by the continuity of p_B and the strong convergence of (T_iS_i) to TS. □

Let E be a (real) Banach lattice. Being regular operators on a Banach lattice, lattice automorphisms of E are automatically bounded, and the group of lattice automorphisms of E equipped with the strong operator topology will be denoted by $\mathrm{Aut}^+(E)$. The subgroup of isometric lattice automorphisms is denoted by $\mathrm{IAut}^+(E)$. Equipped with the strong operator topology, we will denote these spaces by $\mathrm{Aut}_s^+(E)$ and $\mathrm{IAut}_s^+(E)$, and subsets of $\mathrm{Aut}_s^+(E)$ and $\mathrm{IAut}_s^+(E)$ are always assumed to have the strong operator topology.

Definition 4.2.4. Let G be a group and E a Banach lattice. A *positive representation* of G in E is a group homomorphism $\rho\colon G \to \mathrm{Aut}^+(E)$.

For typographical reasons, we will write ρ_s instead of $\rho(s)$, for $s \in G$.

Suppose $\rho\colon G \to \mathrm{Aut}^+(E)$ and $\theta\colon G \to \mathrm{Aut}^+(F)$ are positive representations in the Banach lattices E and F. A positive operator $T\colon E \to F$ is called a *positive intertwiner* of ρ and θ if $T\rho_s = \theta_s T$ for all $s \in G$, and ρ and θ are called *order equivalent* if there exists a positive intertwiner of ρ and θ which is a lattice automorphism. We call them isometrically order equivalent if there exists an intertwiner in $\mathrm{IAut}^+(E)$.

We call a positive representation ρ of G in E *band irreducible* if the only ρ-invariant bands are $\{0\}$ and E. Projection band irreducibility, closed ideal irreducibility, etc., are defined similarly.

In this paper we are, amongst others, concerned with subgroups of $\mathrm{Aut}_s^+(E)$. By the above, $\mathrm{Aut}_s^+(E)$ is a group with a topology such that the multiplication is separately continuous. Lemma 2.2.4 is a useful lemma about such groups, which we state again below.

Lemma 4.2.5. *Let G and H be two groups with a topology such that right multiplication is continuous in both groups, or such that left multiplication is continuous in both groups. Let $\phi\colon G \to H$ be a homomorphism. Then ϕ is continuous if and only if it is continuous at e.*

We continue by examining the center $Z(E)$ of a Banach lattice E, which, as in [32, Definition 3.1.1], is defined to be the set of regular operators m on E satisfying $-\lambda I \leq m \leq \lambda I$ for some $\lambda \geq 0$. With $Z_s(E)$ we denote $Z(E)$ with the strong operator topology. Central operators are often multiplication operators in concrete examples, e.g., if $1 \leq p \leq \infty$ and (Σ, μ) is a finite measure space, then each central operators m on $L^p(\mu)$ is a multiplication operator by an element of $L^\infty(\mu)$, cf. the example following [32, Definition 3.1.1], and this is why we use the notation m for these operators. The center of a Banach lattice is in all respects isomorphic to a space of continuous function, which is the context of the next proposition. For its proof and that of Corollary 4.2.7, we recall some terminology. If E is a Banach lattice, then $\mathrm{Orth}(E)$ denotes the orthomorphisms of E, i.e, the order bounded band preserving operators ([2, Definition 2.41]). An f-algebra is a Riesz space E equipped with a multiplication turning E into an associative algebra, such that if $x, y \in E^+$, then $xy \in E^+$, and if $x \wedge y = 0$, then $xz \wedge y = zx \wedge y = 0$ for all $z \in E^+$ ([2, Definition 2.53]).

Proposition 4.2.6. *Let E be a Banach lattice. Then the center $Z(E)$ equipped with the operator norm is isometrically lattice and algebra isomorphic to the space $C(K)$ for some compact Hausdorff space K, such that the identity operator I is identified with the constant function $\mathbf{1}$.*

Proof. By [32, Theorem 3.1.11], the operator norm of $m \in Z(E)$ is the least $\lambda \geq 0$ such that $-\lambda I \leq m \leq \lambda I$, i.e., it equals the order unit norm corresponding to the order unit I, and so by [32, Proposition 1.2.13] $Z(E)$ is an M-space with order unit I. Then the well-known Kakutani Theorem ([32, Theorem 2.1.3]) yields an isometric lattice isomorphism of $Z(E)$ with a $C(K)$ space such that I corresponds to $\mathbf{1}$. Moreover, by [32, Theorem 3.1.12(ii)] $Z(E) = \mathrm{Orth}(E)$, which is an Archimedean f-algebra by [2, Theorem 2.59]. Clearly $C(K)$ is an Archimedean f-algebra with unit $\mathbf{1}$. By [2, Theorem 2.58] the f-algebra structure on an Archimedean f-algebra is unique, given the positive multiplicative unit, and this implies that the correspondence between $Z(E)$ and $C(K)$ must be an algebra isomorphism. □

This proposition is stated in [44, Proposition 1.4], where a reference to [19] is given for the proof. The development of the theory since the appearance of [19] enables us to give a proof as above.

If E is a Banach lattice, then $\mathrm{ZAut}^+(E)$ denotes the set of central lattice automorphisms, i.e., $\mathrm{ZAut}^+(E) = \mathrm{Aut}^+(E) \cap Z(E)$. Note that $\mathrm{ZAut}^+(E)$ does not denote the center (in the sense of groups) of $\mathrm{Aut}^+(E)$! As before, $\mathrm{ZAut}_s^+(E)$ denotes $\mathrm{ZAut}^+(E)$ equipped with the strong operator topology. In the following corollary we collect a few properties of $\mathrm{ZAut}^+(E)$ as they follow from the isomorphism in Proposition 4.2.6. If K is a compact Hausdorff space, then $C(K)^{++}$ denotes the strictly positive functions in $C(K)$, or equivalently, the positive multiplicatively invertible elements of $C(K)$.

Corollary 4.2.7. *Let E be a Banach lattice. Under the isomorphism $Z(E) \cong C(K)$ from Proposition 4.2.6, we have $\mathrm{ZAut}^+(E) \cong C(K)^{++}$. Consequently, $\mathrm{ZAut}^+(E)$ is a group, and*

$$Z(E) = \mathrm{ZAut}^+(E) - \mathbb{R}^+ \cdot I = \mathrm{ZAut}^+(E) - \mathrm{ZAut}^+(E).$$

Proof. Suppose m corresponds to an element of $C(K)^{++}$. Then m^{-1} corresponds to an element of $C(K)^{++}$ as well, and so m is positive with a positive inverse and hence a lattice automorphism, i.e., $m \in \mathrm{ZAut}^+(E)$. Conversely, let $m \in \mathrm{ZAut}^+(E)$. Then m corresponds to a positive function in $C(K)$. Since $Z(E) = \mathrm{Orth}(E)$, [32, Theorem 3.1.10] shows that, if $m \in Z(E)$ is invertible in $\mathcal{L}(E)$, its inverse is in $Z(E)$ as well. So $m^{-1} \in Z(E) \cong C(K)$, which is only possible if m corresponds to an element of $C(K)^{++}$. The final statement now follows from $C(K) = C(K)^{++} - \mathbb{R}^+ \cdot \mathbf{1}$. □

The next lemma yields an isometric action of the group of lattice automorphisms on the center of a Banach lattice.

Lemma 4.2.8. *Let E be a Banach lattice. Conjugation by elements of* $\mathrm{Aut}^+(E)$ *induces a group homomorphism from* $\mathrm{Aut}^+(E)$ *into the group of isometric algebra and lattice automorphisms of $Z(E)$. If $H \subset \mathrm{Aut}_s^+(E)$ is a uniformly bounded set such that H^{-1} is also uniformly bounded and $A \subset Z_s(E)$ is uniformly bounded, then the map $H \times A \to Z_s(E)$ defined by $(T, m) \mapsto TmT^{-1}$ is continuous. Moreover, if $T \in \mathrm{Aut}^+(E)$ is fixed, then $m \mapsto TmT^{-1}$ is a continuous algebra and lattice automorphism of $Z_s(E)$.*

Proof. Let $T \in \mathrm{Aut}^+(E)$ and $m \in Z(E)$, and take $\lambda \geq 0$. Then

$$\begin{aligned} -\lambda x \leq mx \leq \lambda x \quad \forall x \in E^+ &\Leftrightarrow -\lambda T^{-1}y \leq mT^{-1}y \leq \lambda T^{-1}y \quad \forall y \in E^+ \\ &\Leftrightarrow -\lambda y \leq TmT^{-1}y \leq \lambda y \quad \forall y \in E^+, \end{aligned}$$

hence conjugation by elements of $\mathrm{Aut}^+(E)$ maps $Z(E)$ isometrically into itself. The conjugation action is obviously an algebra automorphism, and if m is positive, then TmT^{-1} is positive as well, so conjugation is positive with a positive inverse, hence a lattice automorphism. The second statement follows from Lemma 4.2.1, and the continuity of $m \mapsto TmT^{-1}$ follows from the separate continuity of multiplication in the strong operator topology. ☐

Finally, we need a proposition for weak integration of strongly continuous center valued functions. If X is a Banach space, (H, dh) a compact Hausdorff probability space, with which we mean a compact Hausdorff space equipped with a not necessarily regular Borel probability measure, and $g \colon H \to X$ a continuous function, then [42, Theorem 3.27] shows that there exists a unique element of X, denoted by $\int_H g(h)\, dh$, defined by duality as follows:

$$\left\langle \int_H g(h)\, dh, x^* \right\rangle = \int_H \langle g(h), x^* \rangle\, dh \quad \forall x^* \in X^*. \tag{4.2.1}$$

Moreover, $\int_H g(h)\, dh \in \overline{\mathrm{co}}(g(H))$. By applying functionals it easily follows that bounded operators can be pulled through the integral, and that the triangle inequality holds.

The above vector valued integral will be used in the next proposition to define an operator valued integral. The Banach space part of the next proposition is a standard argument, which we repeat here for the convenience of the reader.

Proposition 4.2.9. *Let (H, dh) be a compact Hausdorff probability space, let E be a Banach space and let $f \colon H \to \mathcal{L}_s(E)$ be a continuous map. Then the operator $\int_H f(h)\, dh \colon E \to E$, defined by*

$$\left(\int_H f(h)\, dh \right) x := \int_H f(h)x\, dh \quad \forall x \in E,$$

where the second integral is defined by (4.2.1)*, defines an element of $\mathcal{L}(E)$ satisfying $\left\| \int_h f(h)\, dh \right\| \leq \sup_{h \in H} \|f(h)\|$. If $S, T \in \mathcal{L}(E)$, then*

$$S \left(\int_H f(h)\, dh \right) T = \int_H Sf(h)T\, dh. \tag{4.2.2}$$

Moreover, if E is a Banach lattice and $f(H) \subset Z(E)$, then there exist $\lambda, \mu \in \mathbb{R}$ such that $f(H) \subset [\lambda I, \mu I]$, and for such λ and μ we have $\int_H f(h)\, dh \in [\lambda I, \mu I] \subset Z(E)$.

Proof. Note that $f(H)$ is uniformly bounded by the principle of uniform boundedness. The computation

$$\left\| \left(\int_H f(h)\, dh \right) x \right\| = \left\| \int_H f(h)x\, dh \right\| \leq \int_H \|f(h)x\|\, dh \leq \sup_{h \in H} \|f(h)\|\, \|x\|$$

shows that the linear operator $\int_H f(h)\, dh$ is bounded and that its norm satisfies the required estimate. By applying elements of E and functionals, and using the properties of the E-valued integral, (4.2.2) easily follows.

Now assume E is a Banach lattice and $f(H) \subset Z(E)$. By the uniform boundedness of $f(H)$, there exist $\lambda, \mu \in \mathbb{R}$ such that $f(H) \subset [\lambda I, \mu I]$. Suppose λ and μ satisfy this relation, then we have to show that $(\int_H f(h)\, dh)x \in [\lambda x, \mu x]$ for all $x \in E^+$, which is equivalent with

$$\int_H \lambda x\, dh \leq \int_H f(h)x\, dh \leq \int_H \mu x\, dh.$$

Now $f(h)x - \lambda x \in E^+$ for all $h \in H$ by assumption, and since E^+ is a closed convex set, the properties of the E-valued integral imply that $\int_H [f(h)x - \lambda x]\, dh \in E^+$ as well. The second inequality follows similarly. □

4.3 Groups of positive operators

In this section we will relate certain groups of lattice automorphisms to groups of isometric lattice automorphisms. The main assumption on these groups is that every element in the group can be written as a product of a central lattice automorphism and an isometric lattice automorphism. Examples of Banach lattices where this assumption is always satisfied are normalized symmetric Banach sequence spaces, such as c_0 and ℓ^p for $1 \leq p \leq \infty$, where the fact that lattice automorphisms map atoms to atoms easily implies the above property, cf. Section 4.5, and spaces of continuous functions, where there is a well-known characterization of lattice homomorphisms in terms of a multiplication operator and an operator arising from a homeomorphism of the underlying space, cf. Section 4.6. When this assumption is satisfied, we are able to show that there exists a group of isometric lattice automorphisms which has the same invariant ideals as the original group, cf. Theorem 4.3.2. The main result, Theorem 4.3.8, shows that, under the technical Assumption 4.3.3, for every compact group G of lattice automorphisms in which every element can be written as a product of a central lattice automorphism and an isometric lattice automorphisms, there exist a unique compact group H of isometric lattice automorphisms and a non-unique central lattice automorphism m such that $G = mHm^{-1}$.

We start by showing that a certain set of lattice automorphisms is actually a group, and, in fact, a non-trivial semidirect product. Recall that the group of isometric lattice automorphisms of a Banach lattice E is denoted by $\mathrm{IAut}^+(E)$, that the group of central lattice automorphisms of E is denoted by $\mathrm{ZAut}^+(E)$, and that equipped with the strong operator topology these spaces are denoted by $\mathrm{IAut}_s^+(E)$ and $\mathrm{ZAut}_s^+(E)$. The space $\mathrm{ZAut}^+(E)\cdot\mathrm{IAut}^+(E)$ equipped with the strong operator topology is denoted by $\mathrm{ZAut}_s^+(E)\cdot\mathrm{IAut}_s^+(E)$.

Obviously, $\mathrm{IAut}^+(E)$ is a group, and, although not quite so obvious, $\mathrm{ZAut}^+(E)$ is also a group by Corollary 4.2.7. Now suppose $m\phi \in \mathrm{ZAut}^+(E)\cdot\mathrm{IAut}^+(E)$, then $(m\phi)^{-1} = \phi^{-1}m^{-1} = (\phi^{-1}m^{-1}\phi)\phi^{-1}$, which is in $\mathrm{ZAut}^+(E)\cdot\mathrm{IAut}^+(E)$ by Lemma 4.2.8. In a similar vein, if $m_1\phi_1, m_2\phi_2 \in \mathrm{ZAut}^+(E)\cdot\mathrm{IAut}^+(E)$, then $m_1\phi_1 m_2\phi_2 = m_1(\phi_1 m_2\phi_1^{-1})\phi_1\phi_2 \in \mathrm{ZAut}^+(E)\cdot\mathrm{IAut}^+(E)$. Therefore the space $\mathrm{ZAut}^+(E)\cdot\mathrm{IAut}^+(E)$ is a subgroup of $\mathrm{Aut}^+(E)$.

Moreover, the representation of an element $m\phi \in \mathrm{ZAut}^+(E)\cdot\mathrm{IAut}^+(E)$ is unique, and to show this it is sufficient to show that $\mathrm{ZAut}^+(E)\cap\mathrm{IAut}^+(E) = \{I\}$. So suppose $m \in \mathrm{ZAut}^+(E)$ is an isometry. Then $\|m\| = \left\|m^{-1}\right\| = 1$, and taking into account the isometric isomorphism of $Z(E)$ with a $C(K)$ space of Proposition 4.2.6, the continuous function corresponding to m must be unimodular. Since this function is also positive, it must be identically one, and so $m = I$.

By Lemma 4.2.8 the group $\mathrm{IAut}^+(E)$ acts on $\mathrm{ZAut}^+(E)$ by conjugation, and for $\phi \in \mathrm{IAut}^+(E)$ and $m \in \mathrm{ZAut}^+(E)$, this action will be denoted by $\phi(m)$, so $\phi(m) = \phi m\phi^{-1}$. We can form the semidirect product $\mathrm{ZAut}^+(E) \rtimes \mathrm{IAut}^+(E)$, with group operation

$$(m_1, \phi_1)(m_2, \phi_2) := (m_1\phi_1(m_2), \phi_1\phi_2).$$

Using that $\phi(m)$ is the conjugation action of $\phi \in \mathrm{IAut}^+(E)$ on $m \in \mathrm{ZAut}^+(E)$, it easily follows that the map $\chi\colon \mathrm{ZAut}^+(E) \rtimes \mathrm{IAut}^+(E) \to \mathrm{ZAut}^+(E)\cdot\mathrm{IAut}^+(E)$ defined by $\chi(m, \phi) := m\phi$ is a group isomorphism.

All in all, it is now clear that $\mathrm{ZAut}^+(E)\cdot\mathrm{IAut}^+(E)$ is a subgroup of $\mathrm{Aut}^+(E)$, and that it is isomorphic with $\mathrm{ZAut}^+(E) \rtimes \mathrm{IAut}^+(E)$. If necessary we identify $\mathrm{ZAut}^+(E) \rtimes \mathrm{IAut}^+(E)$ and $\mathrm{ZAut}^+(E)\cdot\mathrm{IAut}^+(E)$ through the map χ. The map $p\colon \mathrm{ZAut}^+(E)\cdot\mathrm{IAut}^+(E) \to \mathrm{IAut}^+(E)$ defined by $p(m\phi) := \phi$ is the projection onto the second factor of the semidirect product, which is a group homomorphism.

In the rest of this section we will assume that the group of lattice automorphisms under consideration is contained in $\mathrm{ZAut}^+(E)\cdot\mathrm{IAut}^+(E)$. For certain sequence spaces and spaces of continuous function, we will show that $\mathrm{ZAut}^+(E)\cdot\mathrm{IAut}^+(E)$ equals the whole group of lattice automorphisms, cf. Section 4.5 and Section 4.6, but the next example, which was communicated to us by A.W. Wickstead, shows that there is a simple Banach lattice, not every lattice automorphism of which is a product of a central lattice automorphism and an isometric lattice automorphism.

Example 4.3.1. Consider $\mathbb{R}^2$ with the usual ordering, and with the norm defined by $\|(x, y)\| := \max\{|y|, |x| + |y|/2\}$, so that it becomes a Banach lattice with the standard unit vectors having norm one. Hence $(x, y) \mapsto (y, x)$ is the only possible nontrivial isometric lattice automorphism, but this map is not isometric since $\|(1, 2)\| = 2$ whereas $\|(2, 1)\| = 5/2$. Therefore $(x, y) \mapsto (y, x)$ cannot be a product of a central lattice automorphism and an isometric lattice automorphism, since it is not a central lattice automorphism.

Theorem 4.3.2. *Let E be a Banach lattice and $G \subset \mathrm{ZAut}^+(E) \cdot \mathrm{IAut}^+(E)$ a group. Then $p(G)$ is a group of isometric lattice automorphisms, with the same invariant ideals as G.*

Proof. Let $m \in Z(E)$ and $x \in E^+$. Then there exists a positive λ such that $-\lambda x \leq mx \leq \lambda x$, and so mx is contained in the ideal generated by x. This fact extends to all $x \in E$ by writing $x = x^+ - x^-$, and so m leaves all ideals in E invariant.

Now let $m\phi \in G$, with $m \in \mathrm{ZAut}^+(E)$ and $\phi \in \mathrm{IAut}^+(E)$, let $x \in E$ and let $I \subset E$ be an ideal. Since $m, m^{-1} \in Z(E)$, by the above $x \in I$ if and only if $mx \in I$, and so I is invariant for $m\phi$ if and only if I is invariant for $\phi = p(m\phi)$. $\square$

We will now examine groups $G \subset \mathrm{ZAut}^+(E) \cdot \mathrm{IAut}^+(E)$ which are compact (in the strong operator topology). In this case we can say much more than Theorem 4.3.2, if the following assumption on the Banach lattice is satisfied.

Assumption 4.3.3. If $p \colon \mathrm{ZAut}_s^+(E) \cdot \mathrm{IAut}_s^+(E) \to \mathrm{IAut}_s^+(E)$ denotes the group homomorphism $m\phi \mapsto \phi$, then for any compact subgroup $G \subset \mathrm{ZAut}_s^+(E) \cdot \mathrm{IAut}_s^+(E)$, the map $p|_G$ is continuous.

The next proposition allows us to associate compact subgroups of $\mathrm{IAut}_s^+(E)$ with compact subgroups of $\mathrm{ZAut}_s^+(E) \cdot \mathrm{IAut}_s^+(E)$.

Proposition 4.3.4. *Let E be a Banach lattice satisfying Assumption 4.3.3. Let A be the set of compact subgroups $G \subset \mathrm{ZAut}_s^+(E) \cdot \mathrm{IAut}_s^+(E)$, and let B be the set of pairs (H, q), with $H \subset \mathrm{IAut}_s^+(E)$ a compact subgroup and $q \colon H \to \mathrm{ZAut}_s^+(E) \cdot \mathrm{IAut}_s^+(E)$ a continuous homomorphism such that $p \circ q = \mathrm{id}_H$. Define $\alpha \colon A \to B$ and $\beta \colon B \to A$ by*

$$\alpha(G) := (p(G), (p|_G)^{-1}), \quad \beta(H, q) := q(H).$$

Then for each $G \in A$, $G \mapsto p(G)$ is an isomorphism of compact groups, and α and β are inverses of each other.

Proof. If $G \in A$, then by Assumption 4.3.3 $\ker(p|_G)$ is a compact subgroup of $\mathrm{ZAut}_s^+(E)$, which is isometrically isomorphic with the group $C(K)^{++}$ of strictly positive continuous functions on some compact Hausdorff space K by Corollary 4.2.7. By the principle of uniform boundedness, compact subgroups of $\mathrm{ZAut}_s^+(E)$ are uniformly bounded, and obviously the only uniformly bounded subgroup of $C(K)^{++}$ is trivial, hence $p|_G$ is a group isomorphism. Moreover it is a continuous bijection between a compact space and a Hausdorff space, hence $(p|_G)^{-1}$ is continuous. Clearly $p \circ (p|_G)^{-1} = \mathrm{id}_{p(G)}$, so α is well defined.

Let $G \in A$, then $\beta(\alpha(G)) = \beta(p(G), (p|_G)^{-1}) = G$. Conversely, let $(H, q) \in B$, then $\alpha(\beta(H, q)) = \alpha(q(H)) = (p(q(H)), (p|_{q(H)})^{-1})$, and since $p \circ q = \mathrm{id}_H$ it follows that $p(q(H)) = H$ and that $(p|_{q(H)})^{-1} = (p|_{q(H)})^{-1} \circ p \circ q = q$. $\square$

By the above proposition the compact subgroups $G \subset \mathrm{ZAut}_s^+(E) \cdot \mathrm{IAut}_s^+(E)$ are parametrized by the pairs (H, q) of compact subgroups $H \subset \mathrm{IAut}_s^+(E)$ and continuous homomorphism $q \colon H \to \mathrm{ZAut}_s^+(E) \cdot \mathrm{IAut}_s^+(E)$ satisfying $p \circ q = \mathrm{id}_H$. We will now investigate such maps q, for a given compact subgroup H of $\mathrm{IAut}_s^+(E)$. The condition $p \circ q = \mathrm{id}_H$ is equivalent with the existence of a map $f\colon H \to \mathrm{ZAut}^+(E)$ such that $q(\phi) = f(\phi)\phi$, for $\phi \in \mathrm{IAut}^+(E)$. We now describe the relation between the continuity of f and the continuity of q.

Lemma 4.3.5. *Let E be a Banach lattice satisfying Assumption 4.3.3. Suppose $H \subset \mathrm{IAut}_s^+(E)$ is a compact group and $q\colon H \to \mathrm{ZAut}_s^+(E) \cdot \mathrm{IAut}_s^+(E)$ is a group homomorphism of the form $q(\phi) := f(\phi)\phi$, for some map $f\colon H \to \mathrm{ZAut}_s^+(E)$. Then q is continuous if and only if f is continuous.*

Proof. Suppose f is continuous. Then q is the composition of f and the identity map with the multiplication in $\mathrm{Aut}_s^+(E)$. Since $f(H)$ is compact it is uniformly bounded, and multiplication in the strong operator topology is simultaneously continuous if the first factor is restricted to uniformly bounded sets. Therefore q is continuous.

Conversely, suppose that q is continuous. Then $q(H)$ is a group and it is compact. Moreover $q(H)$ is uniformly bounded, and so $f(H)$, the set of first coordinates of $q(H)$, is also uniformly bounded, since $\|f(\phi)\| = \|f(\phi)\phi\| = \|q(\phi)\|$ for $\phi \in \mathrm{IAut}^+(E)$ by the fact that ϕ is an isometric automorphism. Since the projection onto the second coordinate is continuous on $q(H)$ by Assumption 4.3.3, Lemma 4.2.3 yields the continuity on $q(H)$ of the projection onto the first coordinate. It follows that f is continuous as a composition of q and the projection of $q(H)$ onto the first coordinate. □

We continue describing the structure of maps q as above. For $\phi, \psi \in H$ we have $q(\phi\psi) = f(\phi\psi)\phi\psi$ and

$$q(\phi)q(\psi) = f(\phi)\phi f(\psi)\psi = f(\phi)\phi(f(\psi))\phi\psi.$$

Hence q being a homomorphism is equivalent with $f(\phi\psi) = f(\phi)\phi(f(\psi))$ for all $\phi, \psi \in H$, and such maps are called crossed homomorphisms. We will first show that the image of such crossed homomorphisms is bounded from below.

Lemma 4.3.6. *Let E be a Banach lattice, let $H \subset \mathrm{IAut}_s^+(E)$ be a compact group and let $f\colon H \to \mathrm{ZAut}_s^+(E)$ be a continuous crossed homomorphism, i.e., a continuous map such that $f(\phi\psi) = f(\phi)\phi(f(\psi))$ for all $\phi, \psi \in H$. Then there exists an $\varepsilon > 0$ such that $f(\phi) \geq \varepsilon I$ for all $\phi \in H$.*

Proof. Since $f(H)$ is compact and hence uniformly bounded, there exists some $\lambda > 0$ such that, for all $\phi \in \mathrm{IAut}^+(E)$,

$$\left\|\phi(f(\phi^{-1}))\right\| = \left\|f(\phi^{-1})\right\| \leq \lambda, \tag{4.3.1}$$

since ϕ acts isometrically on $Z(E)$ by Lemma 4.2.8. We identify $\mathrm{ZAut}^+(E)$ with $C(K)^{++}$ for some compact Hausdorff space K, using Corollary 4.2.7. Then (4.3.1) implies

$$0 < \phi(f(\phi^{-1})) \leq \lambda \tag{4.3.2}$$

pointwise on K.

By taking $\phi = \psi = I$ in the definition of a crossed homomorphism, we obtain $f(I) = f(I)f(I)$ and so $\mathbf{1} = f(I)$. For arbitrary $\phi \in \mathrm{IAut}^+(E)$ we obtain

$$\mathbf{1} = f(I) = f(\phi\phi^{-1}) = f(\phi) \cdot \phi(f(\phi^{-1})),$$

and so $f(\phi) \geq 1/\lambda$ pointwise on K by (4.3.2), which establishes the lemma with $\varepsilon = 1/\lambda$. □

To characterize the continuous crossed homomorphisms, we will use the following lemma. It can be viewed as an analytic version of Lemma 3.4.2, which is a standard argument in group cohomology.

Lemma 4.3.7. *Let E be a Banach lattice, and let $H \subset \mathrm{IAut}_s^+(E)$ be a compact group. Let $f\colon H \to \mathrm{ZAut}_s^+(E)$ be a strongly continuous map. Then f is a continuous crossed homomorphism, i.e., a continuous map such that $f(\phi\psi) = f(\phi)\phi(f(\psi))$, where $\phi(m)$ denotes the conjugation action of $\phi \in \mathrm{Aut}^+(E)$ on $m \in \mathrm{ZAut}^+(E)$, if and only if there exists an $m \in \mathrm{ZAut}^+(E)$ such that $f(\phi) = m\phi(m)^{-1}$ for all $\phi \in H$.*

Proof. Suppose f is a continuous crossed homomorphism. The group H is a compact topological group by Corollary 4.2.2, and so we can equip H with its normalized Haar measure $d\psi$. By Proposition 4.2.9 there exist $\lambda, \mu \in \mathbb{R}$ such that $f(H) \subset [\lambda I, \mu I]$ and by Lemma 4.3.6 we may assume that $\lambda > 0$. We use Proposition 4.2.9 to define $m := \int_H f(\psi)\, d\psi$ and also to conclude that this integral is in $[\lambda I, \mu I]$. By Corollary 4.2.7, $[\lambda I, \mu I] \subset \mathrm{ZAut}^+(E)$, and so $m \in \mathrm{ZAut}^+(E)$. Then, for $\phi \in H$, by the left invariance of $d\psi$ and the fact that bounded operators can be pulled through the integral by (4.2.2),

$$\begin{aligned}
\phi(m) &= \phi\left(\int_H f(\psi)\, d\psi\right) \\
&= \int_H \phi(f(\psi))\, d\psi \\
&= \int_H f(\phi)^{-1} f(\phi\psi)\, d\psi \\
&= f(\phi)^{-1} \int_H f(\psi)\, d\psi \\
&= f(\phi)^{-1} m,
\end{aligned}$$

showing that $f(\phi) = m\phi(m)^{-1}$. Conversely, any f defined as above is continuous by Lemma 4.2.8, and such an f is easily seen to be a crossed homomorphism. □

Putting everything together yields the following.

Theorem 4.3.8. *Let E be a Banach lattice satisfying Assumption 4.3.3, and let $G \subset \mathrm{ZAut}_s^+(E) \cdot \mathrm{IAut}_s^+(E)$ be a compact group. Then there exist a unique compact group $H \subset \mathrm{IAut}_s^+(E)$ and an $m \in \mathrm{ZAut}^+(E)$ such that*

$$G = mHm^{-1}.$$

Conversely, if $H \subset \mathrm{IAut}_s^+(E)$ *is a compact subgroup and* $m \in \mathrm{ZAut}^+(E)$*, then* $G \subset \mathrm{ZAut}_s^+(E) \cdot \mathrm{IAut}_s^+(E)$ *defined by the above equation is a compact subgroup of* $\mathrm{ZAut}_s^+(E) \cdot \mathrm{IAut}_s^+(E)$.

Proof. By Proposition 4.3.4, the compact subgroups of $\mathrm{ZAut}_s^+(E) \cdot \mathrm{IAut}_s^+(E)$ are precisely the groups $q(H)$, where H is a compact subgroup of $\mathrm{IAut}_s^+(E)$ and

$$q\colon H \to \mathrm{ZAut}_s^+(E) \cdot \mathrm{IAut}_s^+(E)$$

is a continuous homomorphism satisfying $p \circ q = \mathrm{id}_H$. Using Lemma 4.3.5 and Lemma 4.3.7, we see that the compact subgroups of $\mathrm{ZAut}_s^+(E) \cdot \mathrm{IAut}_s^+(E)$ are precisely the groups of the form $\{m\phi(m)^{-1}\phi : \phi \in H\} = mHm^{-1}$, where H is a compact subgroup of $\mathrm{IAut}_s^+(E)$, and $m \in \mathrm{ZAut}^+(E)$. This establishes the theorem except for the uniqueness of H. As to this, if $G = \{m\phi(m)^{-1}\phi : \phi \in H\}$, for a compact group $H \subset \mathrm{IAut}_s^+(E)$ and $m \in \mathrm{ZAut}^+(E)$, then, in the notation of Proposition 4.3.4, $G = \beta(H, q)$, where $q(\phi) = m\phi(m)^{-1}\phi$ for $\phi \in H$. Since $(H, q) = \alpha(\beta(H, q)) = \alpha(G)$, this implies that $H = p(G)$. Hence H is unique. □

Note that, given a compact subgroup $G \subset \mathrm{ZAut}_s^+(E) \cdot \mathrm{IAut}_s^+(E)$, the compact subgroup $H \subset \mathrm{IAut}_s^+(E)$ is unique, but the element $m \in \mathrm{ZAut}^+(E)$ in Theorem 4.3.8 is obviously not unique, e.g., both m and λm for $\lambda > 0$ generate the same G.

4.4 Positive representations with compact image

In this section we will apply the results from the previous section, in particular Proposition 4.3.4 and Lemma 4.3.7, to representations of groups with compact (in the strong operator topology) image in Banach lattices satisfying Assumption 4.3.3.

Theorem 4.4.1. *Let E be a Banach lattice satisfying Assumption 4.3.3, let G be a group and let* $\rho\colon G \to \mathrm{ZAut}_s^+(E) \cdot \mathrm{IAut}_s^+(E)$ *a positive representation with compact image. Then there exist a unique positive representation* $\pi\colon G \to \mathrm{IAut}_s^+(E)$ *and an* $m \in \mathrm{ZAut}^+(E)$ *such that*

$$\rho_s = m\pi_s m^{-1} \quad \forall s \in G.$$

The image of π is compact. Conversely, a positive representation $\pi\colon G \to \mathrm{IAut}_s^+(E)$ *with compact image and an* $m \in \mathrm{ZAut}^+(E)$ *define a positive representation ρ with compact image in* $\mathrm{ZAut}_s^+(E) \cdot \mathrm{IAut}_s^+(E)$ *by the above equation.*

In this correspondence between ρ and π, ρ is strongly continuous if and only if π is strongly continuous.

Proof. Since $\rho(G)$ is compact, Proposition 4.3.4 applies, and so, combining this with Lemma 4.3.7, $p\colon \rho(G) \to p \circ \rho(G)$ has an inverse of the form $q(\phi) = m\phi(m)^{-1}\phi$ for some $m \in \mathrm{ZAut}^+(E)$ and all $\phi \in p \circ \rho(G)$. We define $\pi := p \circ \rho$, then π has compact image, and for $s \in G$,

$$\rho_s = (q \circ p)(\rho_s) = q(\pi_s) = m\pi_s(m)^{-1}\pi_s = m\pi_s m^{-1}.$$

This shows the existence of π. The uniqueness of π follows from the uniqueness of the factors in $\mathrm{ZAut}^+(E)$ and $\mathrm{IAut}^+(E)$ in

$$\rho_s = m\pi_s m^{-1} = [m\pi_s(m)^{-1}]\pi_s.$$

The remaining statements are now clear. □

Note that, as in Theorem 4.3.8, π is unique, but m is not. Given the positive representation with compact image π, m_1 and m_2 induce the same positive representation with compact image if and only if $m_1^{-1}m_2$ commutes with π_s for all $s \in G$, i.e., if and only if $m_1^{-1}m_2$ intertwines π with itself.

Any representation as in Theorem 4.4.1 is, by that same theorem, obviously order equivalent to an isometric representation. In fact, we can say more. The next proposition has the same proof as Proposition 3.4.6.

Proposition 4.4.2. *Let E be a Banach lattice satisfying Assumption 4.3.3, let G be a group and, using Theorem 4.4.1, let $\rho^1 = m_1\pi^1 m_1^{-1}$ and $\rho^2 = m_2\pi^2 m_2^{-1}$ be positive representations of G with compact image in $\mathrm{ZAut}_s^+(E) \cdot \mathrm{IAut}_s^+(E)$, where π^1 and π^2 are isometric positive representations with compact image in $\mathrm{IAut}_s^+(E)$, and $m^1, m^2 \in \mathrm{ZAut}^+(E)$. Then ρ^1 and ρ^2 are order equivalent if and only if π^1 and π^2 are isometrically order equivalent.*

Proof. In this proof we use semidirect product notation. Suppose that ρ^1 and ρ^2 are order equivalent and let $T = (m, \phi) \in \mathrm{Aut}^+(E)$ be a positive intertwiner. Then, for all $s \in G$,

$$\rho_s^1 T = (m_1\pi_s^1(m_1)^{-1}, \pi_s^1)(m, \phi) = (m_1\pi_s^1(m_1)^{-1}\pi_s^1(m), \pi_s^1\phi) \tag{4.4.1}$$
$$T\rho_s^2 = (m, \phi)(m_2\pi_s^2(m_2)^{-1}, \pi_s^2) = (m\phi(m_2)\phi(\pi_s^2(m_2)^{-1}), \phi\pi_s^2), \tag{4.4.2}$$

and since these are equal, ϕ is a positive isometric intertwiner between π^1 and π^2.

Conversely, let ϕ be a positive isometric intertwiner between π^1 and π^2. Then, by taking $m = m_1\phi(m_2)^{-1}$ and $T = (m, \phi) \in \mathrm{Aut}^+(E)$, it is easily verified that, for all $s \in G$,

$$(m_1\pi_s^1(m_1)^{-1}\pi_s^1(m), \pi_s^1\phi) = (m\phi(m_2)\phi(\pi_s^2(m_2)^{-1}), \phi\pi_s^2),$$

and so, by (4.4.1) and (4.4.2), T intertwines ρ^1 and ρ^2. □

4.5 Positive representations in Banach sequence spaces

In this section we consider positive representations of groups in certain sequence spaces. First we show that every lattice automorphism can be written as a product of a central lattice automorphism and an isometric lattice automorphism. We are also able to show that a large class of sequence spaces satisfy Assumption 4.3.3, and an application of the results from Section 4.3 and Section 4.4 then yields a description of compact groups of lattice automorphisms and of positive representations in these spaces with compact image. Using Theorem 4.3.2, we obtain a decomposition of positive representations into band irreducibles, cf. Theorem 4.5.7. If the representation has compact image, then the irreducible bands in the decomposition in Theorem 4.5.7 are finite dimensional.

We consider normalized symmetric Banach sequence spaces E, by which we mean Banach lattices of sequences equipped with the pointwise ordering and lattice operations such that if $x \in E$ and y is a sequence such that $|y| \leq |x|$, then $y \in E$ and $\|y\| \leq \|x\|$, permutations of sequences in E remain in E with the same norm, and the standard unit vectors $\{e_n\}_{n\in\mathbb{N}}$ have norm 1. Important examples are the classical sequence spaces c_0 and ℓ^p for $1 \leq p \leq \infty$.

If x is a sequence, then $x_{>N}$ denotes the sequence x but with the first N coordinates equal to 0; similarly, $x_{\leq N}$ denotes the sequence x with the coordinates greater than N equal to 0.

We will now show that a normalized symmetric Banach sequence space E satisfies $\mathrm{Aut}^+(E) = \mathrm{ZAut}^+(E) \cdot \mathrm{IAut}^+(E)$. A lattice automorphism must obviously map positive atoms to positive atoms, so for each $T \in \mathrm{Aut}^+(E)$ and $n \in \mathbb{N}$, there exist a unique $m \in \mathbb{N}$ and $\lambda_{mn} > 0$ such that $Te_n = \lambda_{mn} e_m$. Since each $x \in E^+$ is the supremum of the $x_{\leq N}$ for $N \in \mathbb{N}$, the linear span of atoms is order dense, and hence the above relation determines T uniquely. Therefore T can be written as the product of an invertible positive multiplication operator and a permutation operator. We identify the group of permutation operators with $S(\mathbb{N})$, so each $\phi \in S(\mathbb{N})$ corresponds to the operator defined by $(\phi x)_n := x_{\phi^{-1}(n)}$ for $x \in E$ and $n \in \mathbb{N}$. The multiplication operators are identified with ℓ^∞, and by $(\ell^\infty)^{++}$ we denote the set of elements $m \in \ell^\infty$ for which there exists a $\delta > 0$ such that $m_n \geq \delta$ for all $n \in \mathbb{N}$. We conclude that there exist an $m \in (\ell^\infty)^{++}$ and a $\phi \in S(\mathbb{N})$ such that $T = m\phi$. Conversely, an operator defined in this way is a lattice automorphism. Obviously T is a central lattice automorphism if and only if its permutation part is trivial, and so the central lattice automorphisms equal $(\ell^\infty)^{++}$. By Corollary 4.2.7 the center $Z(E)$ equals ℓ^∞. Obviously $S(\mathbb{N}) = \mathrm{IAut}^+(E)$, and so $\mathrm{Aut}^+(E) = (\ell^\infty)^{++} \cdot S(\mathbb{N}) = \mathrm{ZAut}^+(E) \cdot \mathrm{IAut}^+(E)$.

For $\phi \in S(\mathbb{N})$ and $m \in \ell^\infty$, define $\phi(m) \in \ell^\infty$ by $\phi(m)_i := m_{\phi^{-1}(i)}$, the sequence m permuted according to ϕ. Then, for $n \in \mathbb{N}$,

$$\phi m \phi^{-1} e_n = \phi m e_{\phi^{-1}(n)} = \phi m_{\phi^{-1}(n)} e_{\phi^{-1}(n)} = m_{\phi^{-1}(n)} e_n = \phi(m)_n e_n = \phi(m) e_n,$$

which shows that $m \mapsto \phi(m)$ equals the conjugation action of ϕ on m.

We will now show that Assumption 4.3.3 holds, if E has order continuous norm. We will actually show more, namely that $p \colon \mathrm{Aut}^+(E) \to S(\mathbb{N})$ is continuous. The following lemma is a preparation.

Lemma 4.5.1. *Suppose E is a normalized symmetric Banach sequence space. Then the strong operator topology on $S(\mathbb{N}) \subset \mathrm{Aut}^+_s(E)$ is stronger than the topology of pointwise convergence. If E has order continuous norm, then the topologies are equal.*

Proof. If $\phi, \psi, \psi' \in S(\mathbb{N})$ and (ϕ_i) is a net in $S(\mathbb{N})$ converging pointwise to ϕ, then $\psi\phi_i\psi' \to \psi\phi\psi'$ pointwise, so multiplication is separately continuous in the topology of pointwise convergence. We show that the identity map from $S(\mathbb{N})$ equipped with the strong operator topology to $S(\mathbb{N})$ equipped with the topology of pointwise convergence is continuous, and by Lemma 4.2.5 we only have to verify continuity at the identity. Let (ϕ_i) be a net in $S(\mathbb{N})$ converging strongly to the identity, and suppose there is an $n \in \mathbb{N}$ such that $\phi_i(n)$ does not converge to n. If i is such that $\phi_i(n) \neq n$, then $|\phi_i e_n - e_n| \geq e_n$ and so

$$\|\phi_i e_n - e_n\| \geq \|e_n\| = 1.$$

It follows that $\phi_i e_n$ does not converge to e_n, which is a contradiction. This shows that ϕ_i converges pointwise to the identity.

Now suppose that E has order continuous norm. We will show that the identity map from $S(\mathbb{N})$ equipped with the topology of pointwise convergence to $S(\mathbb{N})$ equipped with the strong operator topology is continuous, for which we again only have to verify continuity at the identity. Let (ϕ_i) be a net in $S(\mathbb{N})$ converging pointwise to the identity. Let $x \in E$ and $\varepsilon > 0$. Since $|x|_{>N} \downarrow 0$ for $N \to \infty$, by the order continuity of the norm we can choose N such that $\|x_{>N}\| = \||x|_{>N}\| < \varepsilon/2$. Choose j such that ϕ_i is the identity on all indices $n \leq N$, for all $i \geq j$. Then, for all $i \geq j$,

$$\|\phi_i x - x\| \leq \|\phi_i(x_{>N})\| + \|x_{>N}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

hence ϕ_i converges strongly to the identity. □

This lemma can be used to show that Assumption 4.3.3 holds, if E has order continuous norm.

Lemma 4.5.2. *Let E be a normalized symmetric Banach sequence space with order continuous norm. Then the homomorphism $p\colon \mathrm{Aut}_s^+(E) \to S(\mathbb{N})$ is continuous.*

Proof. Again by Lemma 4.2.5 it suffices to show continuity at the identity. So let $(m_i\phi_i)$ be a net in $\mathrm{Aut}_s^+(E)$ that converges strongly to the identity, and suppose ϕ_i does not converge to the identity. Then by Lemma 4.5.1 there is an $n \in \mathbb{N}$ such that $\phi_i(n)$ does not converge to n, and, for i such that $\phi_i(n) \neq n$, we obtain

$$\|m_i\phi_i e_n - e_n\| = \left\|m_i e_{\phi_i^{-1}(n)} - e_n\right\| \geq \|e_n\| = 1.$$

This contradicts the assumption that $m_i\phi_i$ converges strongly to the identity, and so ϕ_i does converge to the identity. □

Note that ℓ^∞ is a normalized Banach sequence space, and therefore it satisfies $\mathrm{Aut}^+(\ell^\infty) = \mathrm{ZAut}^+(E) \cdot \mathrm{IAut}^+(E)$. It does not have order continuous norm, but it is isometrically lattice isomorphic to $C(K)$ for some compact Hausdorff space K by Kakutani's Theorem, and in Section 4.6 we will show, in Lemma 4.6.3, that such spaces also satisfy Assumption 4.3.3.

Now we can apply the theory of the previous sections, in particular Theorem 4.3.8, Theorem 4.4.1 and Proposition 4.4.2, to obtain the following characterization of compact subgroups of $\mathrm{Aut}_s^+(E)$ and of positive representations with compact image.

Theorem 4.5.3. *Let E be a normalized symmetric Banach sequence space with order continuous norm or ℓ^∞, and let $G \subset \mathrm{Aut}_s^+(E)$ be a compact group. Then there exist a unique compact group $H \subset S(\mathbb{N})$ and an $m \in (\ell^\infty)^{++}$ such that*

$$G = mHm^{-1}.$$

Conversely, if $H \subset S(\mathbb{N})$ is a compact group and $m \in (\ell^\infty)^{++}$, then $G \subset \mathrm{Aut}^+(E)$ defined by the above equation is a compact subgroup of $\mathrm{Aut}_s^+(E)$.

Theorem 4.5.4. *Let E be a normalized symmetric Banach sequence space with order continuous norm or ℓ^∞, let G be a group and let $\rho\colon G \to \mathrm{Aut}_s^+(E)$ be a positive representation with compact image. Then there exist a unique isometric positive representation $\pi\colon G \to S(\mathbb{N})$ and an $m \in (\ell^\infty)^{++}$ such that*

$$\rho_s = m\pi_s m^{-1}, \quad \forall s \in G.$$

The image of π is compact. Conversely, any positive representation $\pi\colon G \to S(\mathbb{N})$ with compact image and $m \in (\ell^\infty)^{++}$ define a positive representation ρ with compact image by the above equation. In this correspondence between ρ and π, ρ is strongly continuous if and only if π is strongly continuous.

Moreover, if $\rho^1 = m_1\pi^1 m_1^{-1}$ and $\rho^2 = m_2\pi^2 m_2^{-1}$ are two positive representations with compact image, where π^1 and π^2 are isometric positive representations with compact image and $m^1, m^2 \in (\ell^\infty)^{++}$, then ρ^1 and ρ^2 are order equivalent if and only if π^1 and π^2 are isometrically order equivalent.

Corollary 4.5.5. *Let E be a normalized symmetric Banach sequence space with order continuous norm or ℓ^∞, and let G be a connected compact group and let $\rho\colon G \to \mathrm{Aut}_s^+(E)$ be a strongly continuous positive representation. Then $\rho_s = I$ for all $s \in G$.*

Proof. We know from Theorem 4.5.4 that $\rho = m\pi m^{-1}$ for some strongly continuous isometric positive representation $\pi\colon G \to S(\mathbb{N})$ and some $m \in (\ell^\infty)^{++}$. For each $n \in \mathbb{N}$, by strong continuity the orbits $\{\pi_s e_n : s \in G\} \subset \{e_m : m \in \mathbb{N}\}$ are connected. Since for $n \neq k$, $\|e_n - e_k\| \geq 1$, the set $\{e_n : n \in \mathbb{N}\}$ is discrete. Hence the orbits consist of one point and π is trivial. But then $\rho = m\pi m^{-1}$ is trivial as well. $\square$

Corollary 4.5.6. *Let E be a normalized symmetric Banach sequence space with order continuous norm or ℓ^∞, and let $G \subset \mathrm{Aut}_s^+(E)$ be a compact group. Then there exists a $\delta > 0$ such that, if $T, S \in G$, $T \neq S$ implies $\|T - S\| \geq \delta$.*

Proof. If $\phi \neq \psi \in S(\mathbb{N})$, then $\|\phi - \psi\| \geq 1$, and so the corollary follows from Theorem 4.5.3. $\square$

In [44, Corollary 3.10], by studying the spectrum of lattice homomorphisms, this corollary is shown with $\delta = \sqrt{3}\sup\{\|T\| : T \in G\}$, for arbitrary uniformly bounded groups of positive operators in complex Banach lattices.

Now we will examine invariant structures under these strongly continuous positive representations. In a Banach lattice with order continuous norm, the collection of bands and the collection of closed ideal coincide by [32, Corollary 2.4.4]. All bands in Banach sequence spaces are of the form $\{x \in E : x_n = 0 \quad \forall n \in \mathbb{N} \setminus A\}$ for some $A \subset \mathbb{N}$; this follows easily from the characterization of bands as disjoint complements. Clearly this collection coincides with the collection of projection bands and the collection of principal bands. We call a series $\sum_{n=1}^{\infty} x_n$ in a Riesz space unconditionally order convergent to x if $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges in order to x for every permutation π of $\mathbb{N}$.

Theorem 4.5.7. *Let E be a normalized symmetric Banach sequence space, let G be a group and let $\rho\colon G \to \mathrm{Aut}^+(E)$ be a positive representation. Then E splits into band irreducibles, in the sense that there exists an α with $1 \leq \alpha \leq \infty$ such that the set of invariant and band irreducible bands $\{B_n\}_{1 \leq n \leq \alpha}$ (if $\alpha < \infty$) or $\{B_n\}_{1 \leq n < \infty}$ (if $\alpha = \infty$) satisfies*

$$x = \sum_{n=1}^{\alpha} P_n x \quad \forall x \in E, \tag{4.5.1}$$

where $P_n\colon E \to B_n$ denotes the band projection, and the series is unconditionally order convergent, hence, in the case that E has order continuous norm, unconditionally convergent.

Moreover, if ρ has compact image and E has order continuous norm or $E = \ell^\infty$, then every invariant and band irreducible band is finite dimensional, and so $\alpha = \infty$.

Proof. We define the isometric positive representation $\pi := p \circ \rho\colon G \to S(\mathbb{N})$, which has the same invariant bands as ρ by Theorem 4.3.2. It follows immediately from the above parametrization of the bands of E that the irreducible bands are given by the orbits $\pi(G)e_n$ of the standard unit vectors e_n. In the case that $\rho(G)$ is compact and E has order continuous norm or $E = \ell^\infty$, the map $p|_{\rho(G)}$ is continuous, so these orbits $p(\rho(G))e_n$ are compact in E, and hence consist of finitely many standard unit vectors, and so the irreducible bands are finite dimensional and there are countable infinitely many of them. The unconditional order convergence of the series (4.5.1) follows from the fact that $|\pi(x)|_{\geq N} \downarrow 0$ as $N \to \infty$ for any permutation π of $\mathbb{N}$. □

In the order continuous case, the series (4.5.1) need not be absolutely convergent, which can be seen by taking the trivial group acting on a normalized symmetric Banach sequence space with order continuous norm not contained in ℓ^1 and taking an $x \in E$ not in ℓ^1.

4.6 Positive representations in $C_0(\Omega)$

In this section we consider the space $C_0(\Omega)$, where Ω is a locally compact Hausdorff space. First we show that every lattice automorphism of $C_0(\Omega)$ is the product of a central lattice automorphism and an isometric lattice automorphism. We will also show that $C_0(\Omega)$ satisfies Assumption 4.3.3, from which we obtain a characterization of compact groups and representations of positive groups with compact image, using the results from Section 4.3 and Section 4.4. As we will explain below, contrary to the sequence space case one cannot expect to find a direct sum type decomposition of an arbitrary strongly continuous positive representation into band irreducibles for general $C_0(\Omega)$. More investigation is necessary to determine whether these representations are still built up, in an appropriate alternative way, from band irreducible representations. As a preparation for such future research, we collect some results about the structure of invariant ideals, bands and projection bands, cf. Proposition 4.6.7.

Analogously to the sequence space case from Section 4.5, we will start by showing that $\mathrm{Aut}^+(C_0(\Omega)) = \mathrm{ZAut}^+(C_0(\Omega)) \cdot \mathrm{IAut}^+(C_0(\Omega))$. Elements of $\mathrm{Homeo}(\Omega)$, the group of homeomorphisms of Ω, are viewed as elements of $\mathrm{Aut}_s^+(C_0(\Omega))$ by $\phi x := x \circ \phi^{-1}$ for $x \in C_0(\Omega)$. The set of multiplication operators by continuous bounded functions is denoted by $C_b(\Omega)$, and by $C_b(\Omega)^{++}$ we denote the elements $m \in C_b(\Omega)$ such that there exists a $\delta > 0$ such that $m(\omega) \geq \delta$ for all $\omega \in \Omega$. It follows from [32, Theorem 3.2.10] that every $T \in \mathrm{Aut}^+(C_0(\Omega))$ can be written uniquely as a product of an element $m \in C_b(\Omega)^{++}$ and an operator $\phi \in \mathrm{Homeo}(\Omega)$, so $T = m\phi$. Conversely, any T defined in this way is a lattice automorphism. It is easy to see that $T \in Z(C_0(\Omega))$ if and only if its part in $\mathrm{Homeo}(\Omega)$ is trivial, so $C_b(\Omega)^{++}$ is the group of central lattice automorphisms, and by Corollary 4.2.7, $Z(C_0(\Omega)) \cong C_b(\Omega)$. Obviously $\mathrm{Homeo}(\Omega) = \mathrm{IAut}^+(E)$, and so $\mathrm{Aut}^+(C_0(\Omega)) = C_b(\Omega)^{++} \cdot \mathrm{Homeo}(\Omega) = \mathrm{ZAut}^+(C_0(\Omega)) \cdot \mathrm{IAut}^+(C_0(\Omega))$.

For $\phi \in \mathrm{Homeo}(\Omega)$ and $m \in C_b(\Omega)^{++}$, define $\phi(m) := m \circ \phi^{-1} \in C_b(\Omega)^{++}$. Then, for $\omega \in \Omega$ and $x \in C_0(\Omega)$,

$$\phi m \phi^{-1} x(\omega) = m\phi^{-1}x(\phi^{-1}(\omega)) = m(\phi^{-1}(\omega))\phi^{-1}x(\phi^{-1}(\omega)) = m(\phi^{-1}(\omega))x(\omega),$$

so $\phi(m)$ equals the conjugation action of ϕ on m.

Our next goal is to show that Assumption 4.3.3 is satisfied, and for that we have to examine $\mathrm{Homeo}(\Omega)$. The topological structure of $\mathrm{Homeo}(\Omega)$ can be described by the following lemma, the proof of which is given by [51, Definition 1.31], [51, Lemma 1.33] and [51, Remark 1.32].

Lemma 4.6.1. *The strong operator topology on* $\mathrm{Homeo}(\Omega)$ *equals the topology with as subbasis elements of the form*

$$U(K, K', V, V') := \{\phi \in \mathrm{Homeo}(\Omega) : \phi(K) \subset V \text{ and } \phi^{-1}(K') \subset V'\}$$

with K and K' compact and V and V' open. A net (ϕ_i) in $\mathrm{Homeo}(\Omega)$ *converges to $\phi \in$* $\mathrm{Homeo}(\Omega)$ *if and only if $\omega_i \to \omega \in \Omega$ implies that $\phi_i(\omega_i) \to \phi(\omega)$ and $\phi_i^{-1}(\omega_i) \to \phi^{-1}(\omega)$.*

Before we can show the validity of Assumption 4.3.3 for $C_0(\Omega)$, we need a small lemma.

Lemma 4.6.2. *Let (m_i) be a net in $C_b(\Omega)^{++}$ and (ϕ_i) be a net in* $\mathrm{Homeo}(\Omega)$ *such that $m_i\phi_i$ converges strongly to the identity. If $\omega_i \to \omega \in \Omega$, then $\phi_i^{-1}(\omega_i) \to \omega$.*

Proof. Suppose that there exists a net (ω_i) converging to ω such that $\phi_i^{-1}(\omega_i)$ does not converge to ω. By passing to a subnet we may assume that there exists an open neighborhood U of x such that $\phi_i^{-1}(\omega_i) \notin U$ for all i. Take a compact neighborhood K of x such that $K \subset U$, then by passing to a subnet we may assume that $\omega_i \in K$ for all i. By [51, Lemma 1.41], a version of Urysohn's Lemma, there exists a function $x \in C_c(\Omega)$ such that x is identically one on K and zero outside of U. Then

$$\begin{aligned}\|m_i\phi_i x - x\| &\geq |[m_i\phi_i x](\omega_i) - x(\omega_i)| \\ &= |m_i(\omega)x(\phi_i^{-1}(\omega)) - x(\omega_i)| = |0 - 1| = 1,\end{aligned}$$

which contradicts the strong convergence of $m_i\phi_i$ to the identity. □

Recall that $p\colon \mathrm{Aut}^+(C_0(\Omega)) \to \mathrm{Homeo}(\Omega)$ denotes the map $m\phi \mapsto \phi$.

Lemma 4.6.3. *Let G be a compact subgroup of $\mathrm{Aut}_s^+(C_0(\Omega))$. Then the map $p|_G\colon G \to \mathrm{Homeo}(\Omega)$ is continuous.*

Proof. Again by Lemma 4.2.5 it is enough to show continuity at the identity. So suppose that (m_i) is a net in $C_b(\Omega)^{++}$ and (ϕ_i) is a net in $\mathrm{Homeo}(\Omega)$ such that $(m_i\phi_i)$ is a net in G converging strongly to the identity. By Lemma 4.2.1 the inverse map in G is continuous, and $(m_i\phi_i)^{-1} = \phi_i^{-1}(m_i^{-1})\phi_i^{-1}$ also converges to the identity. We have to show that ϕ_i converges to the identity in $\mathrm{Homeo}(\Omega)$. Let $\omega_i \to \omega \in \Omega$. By applying Lemma 4.6.2 twice we obtain $\phi_i^{-1}(\omega_i) \to \omega$ and $\phi_i(\omega_i) = (\phi_i^{-1})^{-1}(\omega_i) \to \omega$. This is precisely what we have to show by Lemma 4.6.1. □

Hence Assumption 4.3.3 is satisfied, and once again we can apply the results of Section 4.3 and Section 4.4, in particular Theorem 4.3.8, Theorem 4.4.1 and Proposition 4.4.2, to obtain the following.

Theorem 4.6.4. *Let Ω be a locally compact Hausdorff space and suppose that $G \subset \mathrm{Aut}_s^+(C_0(\Omega))$ is a compact group. Then there exist a unique compact group $H \subset \mathrm{Homeo}(\Omega)$ and an $m \in C_b(\Omega)^{++}$ such that*

$$G = mHm^{-1}.$$

Conversely, any compact group $H \subset \mathrm{Homeo}(\Omega)$ and $m \in C_b(\Omega)^{++}$ define a compact group $G \subset \mathrm{Aut}_s^+(C_0(\Omega))$ by the above equation.

Theorem 4.6.5. *Let Ω be a locally compact Hausdorff space, let G be a group and let $\rho\colon G \to \mathrm{Aut}_s^+(C_0(\Omega))$ be a positive representation with compact image. Then there exist a unique isometric positive representation $\pi\colon G \to \mathrm{Homeo}(\Omega)$ and an $m \in C_b(\Omega)^{++}$ such that*

$$\rho_s = m\pi_s m^{-1} \quad \forall s \in G.$$

The image of π is compact. Conversely, a positive representation $\pi\colon G \to \mathrm{Homeo}(\Omega)$ and an $m \in C_b(\Omega)^{++}$ define a positive representation ρ with compact image by the *above equation. In this correspondence between ρ and π, ρ is strongly continuous if and only if π is strongly continuous.*

Moreover, if $\rho^1 = m_1\pi^1 m_1^{-1}$ and $\rho^2 = m_2\pi^2 m_2^{-1}$ are two positive representations with compact image, where π^1 and π^2 are isometric positive representations with compact image and $m^1, m^2 \in C_b(\Omega)^{++}$, then ρ^1 and ρ^2 are order equivalent if and only if π^1 and π^2 are isometrically order equivalent.

A part of this result is obtained in [11, Example 4.1], by using group cohomology methods on group actions on the set Ω.

Contrary to the sequence space case, our results do not, in general, lead to a decomposition of positive representations with compact image into band irreducibles. Indeed, if the trivial group acts on $C[0,1]$, then every band in $C[0,1]$ is invariant, but since every nonzero band properly contains another nonzero band, there are no invariant band irreducible bands. However, we can still say something about the various invariant structures of such representations, and for this we need a characterization of these structures in $C_0(\Omega)$.

Lemma 4.6.6. *Let Ω be a locally compact Hausdorff space. Every closed ideal $I \subset C_0(\Omega)$ is of the form*

$$I = I_S = \{f \in C_0(\Omega) : f(S) = 0\}$$

for a unique closed $S \subset \Omega$. Hence $S \mapsto I_S$ is an inclusion reversing bijection between the closed subsets of Ω and the closed ideals of $C_0(\Omega)$. The ideal I_S is a band if and only if S is regularly closed, i.e., $S = \mathrm{int}(\overline{S})$, and it is a projection band if and only if S is clopen.

Proof. [32, Proposition 2.1.9] and [32, Corollary 2.1.10] show this statement for Ω compact, and the proof also works if Ω is locally compact. □

The next result may serve as in ingredient in the further study of positive representations in $C_0(\Omega)$. If $\pi \colon G \to \mathrm{Homeo}(\Omega)$ is a map, then a subset $S \subset \Omega$ is called π-invariant if $\pi_s(S) \subset S$ for all $s \in G$.

Proposition 4.6.7. *Let G be a group and let $\rho \colon G \to \mathrm{Aut}^+(C_0(\Omega))$ be a positive representation, and define $\pi := p \circ \rho \colon G \to \mathrm{Homeo}(\Omega)$. Then the map $S \mapsto I_S$ from Lemma 4.6.6 restricts to a bijection between the π-invariant closed subsets $S \subset \Omega$ and the ρ-invariant closed ideals of $C_0(\Omega)$. By further restriction, this induces a bijection between the π-invariant regularly closed subsets of Ω and the ρ-invariant bands of $C_0(\Omega)$, and between the π-invariant clopen subsets of Ω and the ρ-invariant projection bands of $C_0(\Omega)$.*

Proof. By Theorem 4.3.2 the invariant ideals of $\rho(G)$ are the same as the invariant ideals of $\pi(G) \subset \mathrm{Homeo}(\Omega)$, and so we may assume that $\rho = \pi$. Let $S \subset \Omega$ be a π-invariant closed subset. Then, for all $f \in I_S$, $s \in G$ and $\omega \in \Omega$,

$$\pi_s f(\omega) = f(\pi_s^{-1}(\omega)) = 0,$$

and so I_S is a π-invariant closed ideal of $C_0(\Omega)$.

Conversely, let I_S be a π-invariant closed ideal of $C_0(\Omega)$ for some closed $S \subset \Omega$. Let $\omega \in \Omega$, then for all $f \in I_S$ and $s \in G$, $f(\pi_s(\omega)) = \pi_s^{-1} f(\omega) = 0$, and so $\pi_s(\omega) \subset S$, hence S is π-invariant. This shows the statement about closed ideals, and the statements about bands and projection bands follow immediately from Lemma 4.6.6. □

Acknowledgements

The authors thank Anthony Wickstead for providing Example 4.3.1.

Bibliography

[1] C.D. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces with applications to economics*, second ed., American Mathematical Society, Providence, RI, 2003.

[2] ———, *Positive operators*, Springer, Dordrecht, 2006.

[3] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H.P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, and U. Schlotterbeck, *One-parameter semigroups of positive operators*, Springer-Verlag, Berlin, 1986.

[4] W. Beer, *On Morita equivalence of nuclear C^*-algebras*, J. Pure Appl. Algebra **26**.

[5] R.J. Blattner, *Positive definite measures*, Proc. Amer. Math. Soc. (1963), 423–428.

[6] D.P. Blecher, *A Morita theorem for algebras of operators on Hilbert space*, J. Pure Appl. Algebra **156** (2001).

[7] ———, *On Morita's fundamental theorem for C^*-algebras*, Math. Scand. **88** (2001).

[8] D.P. Blecher, P.S. Muhly, and V.I. Paulsen, *Categories of operator modules (Morita equivalence and projective modules)*, Mem. Amer. Math. Soc. **143** (2000).

[9] S. Dirksen, M. de Jeu, and M. Wortel, *Extending representations of normed algebras in Banach spaces*, Operator structures and dynamical systems, Amer. Math. Soc., Providence, RI, 2009, pp. 53–72.

[10] J. Dixmier, *C^*-algebras*, North-Holland Publishing Co., Amsterdam, 1977.

[11] H.A. Dye and R.S. Phillips, *Groups of positive operators*, Canad. J. Math. **8** (1956).

[12] S. Echterhoff, S. Kaliszewski, J. Quigg, and I. Raeburn, *A categorical approach to imprimitivity theorems for C^*-dynamical systems*, Mem. Amer. Math. Soc. **180** (2006).

[13] C. Faith, *Algebra: rings, modules and categories. I*, Springer-Verlag, New York, 1973.

[14] J.M.G. Fell and R.S. Doran, *Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles. Vol. 2*, Academic Press Inc., Boston, MA, 1988.

[15] P. Green, *The local structure of twisted covariance algebras*, Acta Math. **140**.

[16] N. Grønbæk, *Morita equivalence for Banach algebras*, J. Pure Appl. Algebra **99** (1995), no. 2.

[17] ———, *Morita equivalence for self-induced Banach algebras*, Houston J. Math. **22** (1996).

[18] ———, *An imprimitivity theorem for representations of locally compact groups on arbitrary Banach spaces*, Pacific J. Math. **184** (1998).

[19] W. Hackenbroch, *Representation of vector lattices by spaces of real functions*, Functional analysis: surveys and recent results (Proc. Conf., Paderborn, 1976), North-Holland, Amsterdam, 1977, pp. 51–72. North–Holland Math. Studies, Vol. 27.

[20] A.Ya. Helemskii, *Banach and locally convex algebras*, The Clarendon Press Oxford University Press, New York, 1993.

[21] M. de Jeu and H.J.M. Messerschmidt, *A Jordan-Hölder theorem for Riesz spaces*, to appear.

[22] M. de Jeu, H.J.M. Messerschmidt, and M. Wortel, *Crossed products of Banach algebras. II*, to appear.

[23] M. de Jeu and J. Rozendaal, *Decomposing positive representations in L^p-spaces for Polish transformation groups*, to appear, cf. http://www.math.leidenuniv.nl/en/theses/240/.

[24] B.E. Johnson, *Cohomology in Banach algebras*, American Mathematical Society, Providence, R.I., 1972.

[25] S. Lang, *Algebra*, third ed., Springer-Verlag, New York, 2002.

[26] M. Larsen, *On the conjugacy of element-conjugate homomorphisms. II*, Quart. J. Math. Oxford Ser. (2) **47** (1996).

[27] L.H. Loomis, *Positive definite functions and induced representations*, Duke Math. J. **27** (1960).

[28] W.A.J. Luxemburg and A.C. Zaanen, *Riesz spaces. Vol. I*, North-Holland Publishing Co., Amsterdam, 1971.

[29] Yu.I. Lyubich, *Introduction to the theory of Banach representations of groups*, Birkhäuser Verlag, Basel, 1988.

[30] G.W. Mackey, *Imprimitivity for representations of locally compact groups. I*, Proc. Nat. Acad. Sci. U. S. A. **35** (1949).

[31] K. Mansfield, *Induced representations of crossed products by coactions*, J. Funct. Anal. **97** (1991).

[32] P. Meyer-Nieberg, *Banach lattices*, Springer-Verlag, Berlin, 1991.

[33] R.J. Nagel and M. Wolff, *Abstract dynamical systems with an application to operators with discrete spectrum*, Arch. Math. (Basel) **23** (1972).

[34] B. de Pagter and W.J. Ricker, *$C(K)$-representations and R-boundedness*, J. Lond. Math. Soc. (2) **76** (2007), no. 2.

[35] G.K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press Inc., London, 1979.

[36] I. Raeburn and D.P. Williams, *Morita equivalence and continuous-trace C^*-algebras*, American Mathematical Society, Providence, RI, 1998.

[37] M.A. Rieffel, *Induced representations of C^*-algebras*, Bull. Amer. Math. Soc. **78**.

[38] ———, *Morita equivalence for C^*-algebras and W^*-algebras*, J. Pure Appl. Algebra **5**.

[39] ———, *Induced Banach representations of Banach algebras and locally compact groups*, J. Functional Analysis **1** (1967).

[40] ———, *Induced representations of C^*-algebras*, Advances in Math. **13** (1974).

[41] ———, *Morita equivalence for operator algebras*, Operator algebras and applications, Part I (Kingston, Ont., 1980), Amer. Math. Soc., Providence, R.I., 1982, pp. 285–298.

[42] W. Rudin, *Functional analysis*, McGraw-Hill Book Co., New York, 1973.

[43] H.H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, New York, 1974.

[44] H.H. Schaefer, M. Wolff, and W. Arendt, *On lattice isomorphisms with positive real spectrum and groups of positive operators*, Math. Z. **164** (1978), no. 2, 115–123.

[45] J-P. Serre, *Linear representations of finite groups*, Springer-Verlag, New York, 1977.

[46] K. Shiga, *Representations of a compact group on a Banach space*, J. Math. Soc. Japan **7** (1955).

[47] B. de Smit and H.W. Lenstra, Jr., *Linearly equivalent actions of solvable groups*, J. Algebra **228** (2000), no. 1.

[48] L.A. Steen and J.A. Seebach, Jr., *Counterexamples in topology*, Dover Publications Inc., Mineola, NY, 1995.

[49] T. Turumaru, *Crossed product of operator algebra*, Tôhoku Math. J. (2) **10** (1958).

[50] N.R. Wallach, *Real reductive groups. II*, Academic Press Inc., Boston, MA, 1992.

[51] D.P. Williams, *Crossed products of C^*-algebras*, American Mathematical Society, Providence, RI, 2007.

[52] J.A. Wolf, *Harmonic analysis on commutative spaces*, American Mathematical Society, Providence, RI, 2007.

[53] A.C. Zaanen, *Introduction to operator theory in Riesz spaces*, Springer-Verlag, Berlin, 1997.

[54] G. Zeller-Meier, *Produits croisés d'une C^*-algèbre par un groupe d'automorphismes*, J. Math. Pures Appl. (9) **47** (1968).

Dankwoord

Op de eerste plaats wil ik mijn begeleider Marcel de Jeu bedanken voor zijn inzet en onze prettige samenwerking, in het bijzonder voor zijn grote hoeveelheid nauwkeurig commentaar op mijn schrijfstijl en zijn vele nuttige en wiskundig precieze opmerkingen en toevoegingen. Ook wil ik graag al mijn collega's van het Mathematisch Instituut bedanken, in het bijzonder mijn medepromovendus Miek Messerschmidt, met wie ik vele interessante discussies heb gevoerd, en Lenny Taelman, voor zijn commentaar over groepen van eindigdimensionale positive operatoren. Ook wil ik graag mijn ouders bedanken voor hun steun gedurende mijn studie en promotie.

Samenvatting

Gemotiveerd door onder andere de quantummechanica zijn sterk continue unitaire representaties van groepen intensief bestudeerd. Voor zulke representaties van compacte groepen is er een decompositie in eindigdimensionale irreducibele representaties als orthogonale directe som mogelijk. Als de groep niet compact is en de representatieruimte een separabele Hilbertruimte is, is er nog steeds een decompositie in irreducibelen in termen van directe integralen van Hilbertruimtes en directe integralen van representaties.

Als een groep werkt op een verzameling induceert dat vaak een unitaire representatie in een ruimte van L^2-functies. In dergelijke situaties krijgen we dan meestal ook sterk continue representaties in L^p-ruimtes en ruimtes van continue functies. Dit zijn representaties in Banachruimtes die geen unitaire representaties zijn. Toch willen we graag dit soort representaties begrijpen. In het bijzonder, is er voor dit soort representaties ook een decompositie in irreducibelen mogelijk?

Dit proefschrift is een bijdrage aan de theorie van zulke representaties. Het bestaat uit twee delen. In het eerste gedeelte bekijken we gekruiste producten, en in het tweede deel bekijken we positieve representaties. We beschouwen nu eerst het eerste gedeelte.

In de representatietheorie zijn er, gegeven een groep, vaak algebra's beschikbaar met de eigenschap dat representaties van de groep in bijectie zijn met representaties van de algebra. Een voorbeeld is de groepsalgebra van een eindige groep, waarvan representaties in een vectorruimte in bijectie zijn met representaties van de groep in die vectorruimte. In de unitaire theorie is er ook zo'n algebra, de groep C^*-algebra, waarvan de zogenaamde niet-degenereerde representaties in een Hilbertruimte in bijectie zijn met unitaire representaties van de groep in die Hilbertruimte. Deze algebra is erg nuttig gebleken in de decompositietheorie van unitaire representaties.

Dit is de reden dat we, gegeven een lokaal compacte groep, ook een algebra tot onze beschikking willen hebben waarvan een klasse van zijn representaties in Banachruimtes in bijectie is met een klasse van sterk continue representaties van de groep. De reden dat we niet alle representaties beschouwen is als volgt. Er is maar één oneindigdimensionale separabele Hilbertruimte op isomorfie na, namelijk ℓ^2, terwijl er een grote diversiteit aan oneindigdimensionale separabele Banachruimtes is. De hierboven genoemde algemene decompositiestelling is dus in feite een stelling over één ruimte. Het is te lastig om alle representaties in alle Banachruimtes te bekijken. Het zou beter zijn om voor specifieke klassen van representaties specifieke

Banachalgebra's te hebben met de juiste bijectie-eigenschap. Met andere woorden, de constructie van de groep C^*-algebra moet gegeneraliseerd worden, zodanig dat deze aangepast kan worden aan de klasse representaties waarin men geïnteresseerd is, in plaats van alleen voor unitaire representaties te werken.

Dit is in Hoofdstuk 2 gedaan, waarin we een nog algemener object dan de groep C^*-algebra generaliseren, namelijk de gekruist product C^*-algebra. Gegeven een C^*-dynamisch systeem is de gekruist product C^*-algebra een C^*-algebra, waarvan de niet-gedegenereerde representaties in bijectie zijn met de zogenaamde covariante representaties van het C^*-dynamische systeem. In ons geval kijken we naar een Banachalgebra dynamisch systeem, en zoals hierboven uitgelegd is, is een nieuwe variabele hierbij een klasse $\mathcal{R}$ van covariante representaties, die in het C^*-geval gelijk is aan alle covariante representaties. Het hoofdresultaat uit dit hoofdstuk laat zien dat er een Banachalgebra bestaat, de gekruist product Banachalgebra, waarvan de niet-gedegenereerde algebrarepresentaties in bijectie zijn met de zogenaamde $\mathcal{R}$-continue covariante representaties van het oorspronkelijke Banachalgebra dynamische systeem, een in het algemeen grotere klasse dan $\mathcal{R}$. In het C^*-geval vallen deze klassen samen. Als we nu een Banachalgebra dynamisch systeem nemen met een triviale Banachalgebra, dan krijgen we een algebra waarvan de niet-gedegenereerde representaties in bijectie zijn met een klasse van representaties van een lokaal compacte groep.

Dit hebben we toegepast op specifieke situaties, waardoor we nu klassen van groepsrepresentaties in Banachruimtes kunnen bekijken door de algebrarepresentaties van het corresponderende Banachalgebra gekruist product te bestuderen (het passende analogon van de groep C^*-algebra), die in principe makkelijker te begrijpen zijn aangezien Banachalgebra's veel meer functionaalanalytische structuur hebben dan groepen.

In Hoofdstuk 3 en Hoofdstuk 4 bestuderen we positieve representaties in Rieszruimtes en Banachroosters, waarbij we in het bijzonder geïnteresseerd zijn in decomposities in irreducibelen. Veel representaties uit de praktijk, bijvoorbeeld representaties geïnduceerd door groepsacties op verzamelingen, zijn positieve representaties. In Hoofdstuk 3 bekijken we het eenvoudigste geval: eindige groepen.

De eerste vraag is wat het juiste concept van irreducibiliteit is in de geordende context. In de unitaire theorie is irreducibiliteit van representaties equivalent met indecomposabiliteit, waarmee bedoeld wordt dat de representatie niet orthogonaal opgesplitst kan worden in twee deelruimtes die invariant zijn onder de representatie. In onze geordende context blijkt dat orde indecomposabiliteit van representaties te zijn, d.w.z. dat de ruimte niet geordend opgesplitst kan worden in twee deelruimtes die invariant zijn onder de representatie.

We zijn dus in het bijzonder geïnteresseerd in orde indecomposabele representaties, en een natuurlijke vraag is of dergelijke representaties van eindige groepen altijd eindigdimensionaal zijn. In de unitaire theorie is de analoge uitspraak altijd waar en het bewijs is eenvoudig. Dit unitaire bewijs werkt echter niet in de geordende context. Eén van de hoofdresultaten van Hoofdstuk 3 laat echter zien dat ook in de geordende context het antwoord op deze vraag altijd bevestigend is, mits de ruimte waarin we kijken Dedekind volledig is. Het bewijs gebruikt inductie naar de orde van de groep en toont overigens aanzienlijk meer aan dan we hier noemen.

In dit hoofdstuk karakteriseren we ook de groep van roosterautomorfismen van $\mathbb{R}^n$ met de kanonieke ordening, voor $n \in \mathbb{N}$. Alle eindigdimensionale Archimedische Rieszruimtes zijn isomorf met $\mathbb{R}^n$ voor een $n \in \mathbb{N}$. Deze groep blijkt een semidirect product te zijn van de groep van diagonaalmatrices met strikt positieve diagonaalelementen en de groep van permutatiematrices. Hieruit volgt uiteindelijk dat alle eindige groepen van roosterautomorfismen groepen van permutatiematrices zijn, geconjugeerd door een diagonaalmatrix met strikt positieve diagonaalelementen. Dit impliceert dat representaties van eindige groepen in eindigdimensionale Archimedische Rieszruimtes op een soortgelijke manier beschreven kunnen worden. Uiteindelijk verkrijgen we een beschrijving van de ordeduaal van een eindige groep, d.w.z. van de equivalentieklassen van orde-equivalente orde indecomposabele positieve representaties in Dedekind volledige Rieszruimtes, in termen van groepsacties op eindige verzamelingen. We laten ook zien dat een positieve representatie van een eindige groep in een eindigdimensionale Archimedische Rieszruimte altijd uniek opsplitst in orde indecomposabele representaties.

Het blijkt verder dat er voorbeelden zijn van orde equivalente orde indecomposabele representaties met hetzelfde karakter, dus karakters bepalen geen positieve representaties zoals in het unitaire geval. Ook hebben we inductie in het geordende geval bekeken, waarvan de categorietheoretische aspecten grotendeels hetzelfde zijn als in het niet geordende geval. De multipliciteitsversie van Frobenius reciprociteit blijkt echter niet te gelden.

In Hoofdstuk 4 nemen we de volgende stap: na de eindige groepen gaan we over op sterk continue positieve representaties van compacte groepen. Om deze te bestuderen bekijken we groepen van roosterautomorfismen die compact zijn in de sterke operatortopologie. We nemen aan dat deze groepen bevat zijn in het product, dat weer een semidirect product is, van de groep van centrale roosterautomorfismen en de groep van isometrische roosterautomorfismen. Dit is gemotiveerd door het hierboven beschreven semidirect product in het eindigdimensionale geval. Voor genormaliseerde symmetrische Banach rijtjesruimtes met orde continue norm en voor ruimtes van continue functies geldt zelfs dat de hele groep van roosterautomorfismen gelijk is aan dit semidirecte product, en dus geldt bovenstaande aanname voor compacte groepen van roosterautomorfismen altijd.

Onder een milde technische aanname die geldt in bovenstaande rijtjesruimtes en ruimtes van continue functies blijkt weer dat zulke compacte groepen van roosterautomorfismen gelijk zijn aan groepen van isometrische roosterautomorfismen, geconjugeerd met een centraal roosterautomorfisme. Dit levert weer een goede beschrijving op van sterk continue positieve representaties van compacte groepen, en als we dit toepassen op rijtjesruimtes, krijgen we een resultaat analoog aan het unitaire geval in het begin van deze samenvatting: elke sterk continue positieve representatie van een compact groep in een genormaliseerde symmetrische Banach rijtjesruimte met orde continue norm of in ℓ^∞ heeft een decompositie als orde directe som van eindigdimensionale orde indecomposabele deelrepresentaties. Deze klasse van rijtjesruimtes bevat de klassieke rijtjesruimtes c_0 en ℓ^p voor $1 \leq p < \infty$.

Curriculum Vitae

Marten Wortel werd op 15 juli 1981 geboren in Delft, waar hij in 1999 zijn gymnasium diploma behaalde aan het Stanislascollege. In datzelfde jaar begon hij de studie technische wiskunde in Delft. Eind 2007 studeerde hij daar af, met de masterscriptie *Vector-valued Function Spaces and Tensor Products* onder begeleiding van prof. dr. B. de Pagter. Vervolgens begon hij in december 2007 als Assistent in Opleiding aan het Mathematisch Instituut in Leiden bij prof. dr. S.M. Verduyn Lunel onder dagelijkse begeleiding van dr. M.F.E. de Jeu. Hij bezocht conferenties in Leiden en Kopenhagen, en in Madrid, Dresden, Tianjin en Athene bezocht hij conferenties waar hij ook voordrachten hield over zijn werk. Dit proefschrift is het resultaat van zijn promotieonderzoek.